AN ELEMENTARY TREATISE

ON

HYDRODYNAMICS AND SOUND

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BY

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PREFACE.

THE treatise on Hydrodynamics, which I published in 1888, was intended for the use of those who are acquainted with the higher branches of mathematics, and its aim was to present to the reader as comprehensive an account of the whole subject as was possible. But although a somewhat formidable battery of mathematical artillery is indispensable to those who desire to possess an exhaustive knowledge of any branch of mathematical physics, yet there are a variety of interesting and important investigations, not only in Hydrodynamics but also in Electricity and other physical subjects, which are well within the reach of every one who possesses a knowledge of the elements of the Differential and Integral Calculus and the fundamental principles of Dynamics. I have accordingly, in the present work, abstained from introducing any of the more advanced methods of analysis, such as Spherical Harmonics, Elliptic Functions and the like; and, as regards the dynamical portion of the subject, I have endeavoured to solve the various problems which present themselves, by the aid of the Principles of Energy and Momentum, and have avoided the use of Lagrange's equations. There are a few problems, such as the helicoidal steady motion and stability of a solid of revolution moving in an infinite liquid, which cannot be conveniently treated without having recourse to moving axes; but as the theory of moving axes is not an altogether easy branch of Dynamics, I have as far as possible abstained from introducing

them, and the reader who is unacquainted with the use of moving axes is recommended to omit those sections in which they are employed.

The present work is principally designed for those who are reading for the Mathematical Tripos and for other examinations in which an elementary knowledge of Hydrodynamics and Sound is required; but I also trust that it will not only be of service to those who have neither the time nor the inclination to become conversant with the intricacies of the higher mathematics, but that it will also prepare the way for the acquisition of more elaborate knowledge, on the part of those who have an opportunity of devoting their attention to the more recondite portions of these subjects.

The first part, which relates to Hydrodynamics, has been taken with certain alterations and additions from my larger treatise, and the analytical treatment has been simplified as much as possible. I have thought it advisable to devote a chapter to the discussion of the motion of circular cylinders and spheres, in which the equations of motion are obtained by the direct method of calculating the resultant pressure exerted by the liquid upon the solid; inasmuch as this method is far more elementary, and does not necessitate the use of Green's Theorem, nor involve any further knowledge of Dynamics on the part of the reader than the ordinary equations of motion of a rigid body. The methods of this chapter can also be employed to solve the analogous problem of determining the electrostatic potential of cylindrical and spherical conductors and accumulators, and the distribution of electricity upon such surfaces. The theory of the motion of a solid body and the surrounding liquid, regarded as a single dynamical system, is explained in Chapter III., and the motion of an elliptic cylinder in an infinite liquid, and the motion of a circular cylinder in a liquid bounded by a rigid plane are discussed at length.

The Chapters on Waves and on Rectilinear Vortex Motion comprise the principal problems which admit of treatment by elementary methods, and I have also included an investigation

due to Lord Rayleigh, respecting one of the simpler cases of the instability of fluid motion.

In the second part, which deals with the Theory of Sound, I have to acknowledge the great assistance which I have received from Lord Rayleigh's classical treatise. This part contains the solution of the simpler problems respecting the vibrations of strings, membranes, wires and gases. A few sections are also devoted to the Thermodynamics of perfect gases, principally for the sake of supplementing Maxwell's treatise on Heat, by giving a proof of some results which require the use of the Differential Calculus.

The present edition has been carefully revised throughout, and a certain amount of new matter has been added. I have devoted Chapter IX. to the flexion and vibrations of naturally straight wires and rods; whilst an entirely new chapter has been added on the finite deformation of naturally straight and curved wires, in which I have discussed a variety of questions which admit of fairly simple mathematical treatment.

FLEDBOROUGH HALL,
HOLYPORT, BERKS.

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PART I.

HYDRODYNAMICS.

CHAPTER I.

ON THE EQUATIONS OF MOTION OF A PERFECT FLUID.

1. THE object of the science of Hydrodynamics is to investigate the motion of fluids. All fluids with which we are acquainted may be divided into two classes, viz. incompressible fluids or liquids, and compressible fluids or gases. It must however be recollected that all liquids experience a slight compression, when submitted to a sufficiently large pressure, and therefore in strictness a liquid cannot be regarded as an incompressible fluid; but inasmuch as the compression produced by such pressures as ordinarily occur is very small, liquids may be usually treated as incompressible fluids without sensible error. The physical interest arising from the study of the motion of gases, is due to the fact that air is the vehicle by means of which sound is transmitted. We shall therefore devote the first part of this volume to the discussion of incompressible fluids or liquids, reserving the discussion of gases for the second part, which deals with the Theory of Sound.

We must now define a fluid.

2. *A fluid may be defined to be an aggregation of molecules, which yield to the slightest effort made to separate them from each other, if it be continued long enough.*

A *perfect* fluid, is one which is incapable of sustaining any tangential stress or action in the nature of a shear; and it will be shown in § 15 that the consequence of this property is, that the pressure at every point of a perfect fluid is equal in all directions, whether the fluid be at rest or in motion. A perfect fluid is however an entirely ideal substance, since all fluids

with which we are acquainted are capable of offering resistance to tangential stress. This property, which is known as viscosity, gives rise to an action in the nature of friction, by which kinetic energy is gradually converted into heat.

In the case of gases, water and many other liquids, the effect of viscosity is so small that such fluids may be approximately regarded as perfect fluids. The neglect of viscosity very much simplifies the mathematical treatment of the subject, and in the present treatise, we shall confine our attention to perfect fluids.

Before entering upon the dynamical portion of the subject, it will be convenient to investigate certain kinematical propositions, which are true for all fluids.

Kinematical Theorems.

3. The motion of a fluid may be investigated by two different methods, the first of which is called the Lagrangian method, and the second the Eulerian or flux method, although both are due to Euler.

In the Lagrangian method, we fix our attention upon an element of fluid, and follow its motion throughout its history. The variables in this case are the initial coordinates a, b, c of the particular element upon which we fix our attention, and the time. This method has been successfully employed in the solution of very few problems.

In the Eulerian or flux method, we fix our attention upon a particular point of the space occupied by the fluid, and observe what is going on there. The variables in this case are the coordinates x, y, z of the particular point of space upon which we fix our attention, and the time.

Velocity and Acceleration.

4. In forming expressions for the velocity and acceleration of a fluid, it is necessary to carefully distinguish between the Lagrangian and the flux method.

I. *The Lagrangian Method.*

Let u, v, w be the component velocities parallel to *fixed* axes, of an element of fluid whose coordinates are x, y, z and $x + \delta x$, $y + \delta y$, $z + \delta z$ at times t and $t + \delta t$ respectively, then

$$u = dx/dt = \dot{x}, \quad v = \dot{y}, \quad w = \dot{z} \qquad (1),$$

where in forming $\dot{x}$, $\dot{y}$, $\dot{z}$ we must suppose x, y, z to be expressed in terms of the initial coordinates a, b, c and the time.

The expressions for the component accelerations are

$$f_x = \dot{u} = \ddot{x},\ f_y = \ddot{y},\ f_z = \ddot{z} \quad\text{..................}\quad (2),$$

where u, v, w are supposed to be expressed in terms of a, b, c and t.

II. *The Flux Method.*

5. Let δQ be the quantity of fluid which in time δt flows across any small area A, which passes through a fixed point P in the fluid; let ρ be the density of the fluid, q its resultant velocity, and ϵ the angle which the direction of q makes with the normal to A drawn towards the direction in which the fluid flows. Then

$$\delta Q = \rho q A \delta t \cos\epsilon,$$

therefore

$$q = \frac{1}{\rho A \cos\epsilon} \frac{dQ}{dt}.$$

Now $A\cos\epsilon$ is the projection of A upon a plane passing through P perpendicular to the direction of motion of the fluid; hence δQ is independent of the direction of the area, and is the same for all areas whose projections upon the above-mentioned plane are equal. Hence the velocity is equal to the rate per unit of area divided by the density, at which fluid flows across a plane perpendicular to its direction of motion.

The velocity is therefore a function of the position of P and the time.

In the present treatise the flux method will almost exclusively be employed. We may therefore put $u = F(x, y, z, t)$; whence if $u + \delta u$ be the velocity parallel to x at time $t + \delta t$ of the element of fluid which at time t was situated at the point (x, y, z),

$$\delta u = F(x + u\delta t,\ y + v\delta t,\ z + w\delta t,\ t + \delta t) - F(x, y, z, t).$$

Therefore the acceleration,

$$f_x = \lim \frac{\delta u}{\delta t} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}.$$

Hence if $\partial/\partial t$ denotes the operator

$$d/dt + ud/dx + vd/dy + wd/dz,$$

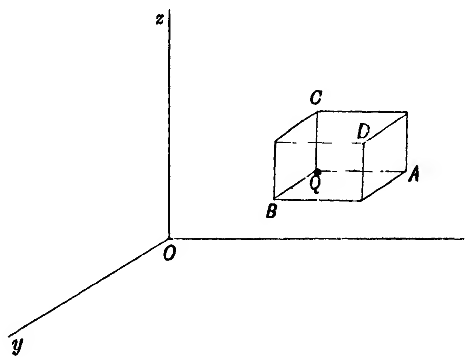
the component accelerations will be given by the equations

$$f_x = \frac{\partial u}{\partial t},\ f_y = \frac{\partial v}{\partial t},\ f_z = \frac{\partial w}{\partial t} \quad\text{..................}\quad (3).$$

The Equation of Continuity.

6. If an imaginary fixed closed surface be described in a fluid, the difference between the amounts of fluid which flow in and flow out during a small interval of time δt, must be equal to the increase in the amount of fluid during the same interval, which the surface contains.

The analytical expression for this fact is called *the equation of continuity.*



Let Q be any point (x, y, z), and consider an elementary parallelopiped $\delta x \delta y \delta z$.

The amount of fluid which flows in across the face CB in time δt is

$$\rho u \delta y \delta z \delta t.$$

The amount which flows out across the opposite face AD is

$$\rho u \delta y \delta z \delta t + \frac{d}{dx}(\rho u)\, \delta x \delta y \delta z \delta t,$$

whence the gain of fluid due to the fluxes across the faces CB, AD is

$$-\frac{d}{dx}(\rho u)\, \delta x \delta y \delta z \delta t.$$

Treating the other faces in a precisely similar manner, it follows that the total gain is

$$-\left\{\frac{d}{dx}(\rho u) + \frac{d}{dy}(\rho v) + \frac{d}{dz}(\rho w)\right\} \delta x \delta y \delta z \delta t \quad \ldots\ldots\ldots (4).$$

The amount of fluid within the element at time t is $\rho\,\delta x\,\delta y\,\delta z$, and therefore the amount at time $t+\delta t$ is

$$\left(\rho + \frac{d\rho}{dt}\,\delta t\right)\delta x\,\delta y\,\delta z.$$

The gain is therefore

$$\frac{d\rho}{dt}\,\delta x\,\delta y\,\delta z\,\delta t.$$

Equating this to (4) we obtain the equation

$$\frac{d\rho}{dt} + \frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} = 0 \quad \text{...........(5)}.$$

This equation is called the *equation of continuity.*

In the case of a liquid, ρ ~~is~~ constant, ~~and~~ (5) takes the simple form

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad \text{....................(6)}.$$

We shall hereafter require the equation of continuity of a liquid referred to polar coordinates. This may be obtained in a similar manner by considering a polar element of volume $r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\omega$, and it can be shown that if u, v, w be the velocities in the directions in which r, θ, ω increase, the required equation is

$$\sin\theta\,\frac{d(r^2 u)}{dr} + r\,\frac{d(v\sin\theta)}{d\theta} + r\,\frac{dw}{d\omega} = 0 \quad \text{........(7)}.$$

If ϖ, θ, z be cylindrical coordinates, the equation is

$$\frac{d(\varpi u)}{d\varpi} + \frac{dv}{d\theta} + \varpi\,\frac{dw}{dz} = 0 \quad \text{.................(8)}.$$

The Velocity Potential.

7. In a large and important class of problems, the quantity $udx + vdy + wdz$ is a perfect differential of a function of x, y, z which we shall call ϕ; when this is the case, we shall have

$$udx + vdy + wdz = d\phi,$$

whence
$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz} \quad \text{..............(9)}.$$

Substituting these values of u, v, w in (6) we obtain

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0 \quad \text{....................(10)},$$

or as it is usually written

$$\nabla^2\phi = 0$$

This equation is called *Laplace's equation*, from the name of its discoverer; it is a very important equation, which continually occurs in a variety of branches of physics. The operator ∇^2 is called *Laplace's operator*.

We can now obtain the transformation of Laplace's equation when polar coordinates are employed. For in this case

$$udr + vrd\theta + wr\sin\theta d\omega = d\phi,$$

whence $$u = \frac{d\phi}{dr}, \quad v = \frac{1}{r}\frac{d\phi}{d\theta}, \quad w = \frac{1}{r\sin\theta}\frac{d\phi}{d\omega} \quad \text{...........(11)}.$$

Substituting in (7) we obtain

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2\phi}{d\omega^2} = 0 \quad \text{..(12)}.$$

The equation of continuity and the theory of the velocity potential may therefore be employed to effect transformations, which it would be very laborious to work out by the usual methods for the change of the independent variables.

8. The existence of a velocity potential involves the conditions that each of the three quantities

$$\frac{dw}{dy} - \frac{dv}{dz}, \quad \frac{du}{dz} - \frac{dw}{dx}, \quad \frac{dv}{dx} - \frac{du}{dy}$$

should be zero; when such is not the case we shall denote these quantities by 2ξ, 2η, 2ζ. The quantities ξ, η, ζ, for reasons which will be explained hereafter, are called *components of molecular rotation*, they evidently satisfy the equation

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0.$$

When a velocity potential exists, the motion is called *irrotational*; and when a velocity potential does not exist, the motion is called *rotational or vortex motion.*

Lines of Flow and Stream Lines.

9. Def. A *line of flow* is a line whose direction coincides with the direction of the resultant velocity of the fluid.

The differential equations of a line of flow are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Hence if $\chi_1(x, y, z, t) = \alpha_1$, $\chi_2(x, y, z, t) = \alpha_2$ be any two independent integrals, the equations $\chi_1 = \text{const.}$, $\chi_2 = \text{const.}$, are the equations of two families of surfaces whose intersections determine the lines of flow.

DEF. A *stream line or a line of motion*, is a line whose direction coincides with the direction of the actual paths of the elements of fluid.

The equations of a stream line are determined by the simultaneous differential equations,

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = w,$$

where x, y, z must be regarded as unknown functions of t. The integration of these equations will determine x, y, z in terms of the initial coordinates and the time.

10. When a velocity potential exists, the equation

$$udx + vdy + wdz = 0$$

is the equation of a family of surfaces, at every point of which the velocity potential has a definite constant value, and which may be called *surfaces of equi-velocity potential.*

If P be any point on the surface, $\phi = \text{const.}$, and dn be an element of the normal at P which meets the neighbouring surface $\phi + \delta\phi$ at Q, the velocity at P along PQ, will be equal to $d\phi/dn$; hence $d\phi$ must be positive, and therefore a fluid always flows from places of lower to places of higher velocity potential.

The lines of flow evidently cut the surfaces of equi-velocity potential at right angles.

11. The solution of hydrodynamical problems is much simplified by the use of the velocity potential (whenever one exists), since it enables us to express the velocities in terms of a single function ϕ. But when a velocity potential does not exist, this cannot in general be done, unless the motion either takes place in two dimensions, or is symmetrical with respect to an axis.

In the case of a liquid, if the motion takes place in planes parallel to the plane of xy, the equation of the lines of flow is

$$udy - vdx = 0 \quad \text{......................(13).}$$

The equation of continuity is

$$\frac{du}{dx} + \frac{dv}{dy} = 0,$$

which shows that the left-hand side of (13) is a perfect differential $d\psi$, whence

$$u = \frac{d\psi}{dy}, \quad v = -\frac{d\psi}{dx} \quad \text{.................(14).}$$

The function ψ is called Earnshaw's current function.

When the motion takes place in planes passing through the axis of z, the equation of the lines of flow may be written

$$\varpi\,(w d\varpi - u dz) = 0 \quad \text{....................(15),}$$

where ϖ, z are cylindrical coordinates.

By (8) the equation of continuity is

$$\frac{d(\varpi u)}{d\varpi} + \varpi\frac{dw}{dz} = 0,$$

which shows that the left-hand side of (15) is a perfect differential $d\psi$, whence

$$w = \frac{1}{\varpi}\frac{d\psi}{d\varpi}, \quad u = -\frac{1}{\varpi}\frac{d\psi}{dz} \quad \text{.................(16).}$$

The function ψ is called Stokes' current function.

The Bounding Surface.

12. Besides the equations which must be satisfied within the interior of a fluid, it is necessary that certain other conditions should be satisfied at the boundary, which depend upon the special problem under consideration.

If the fluid is bounded by a surface, whose equation referred to axes fixed in space is $F(x, y, z, t) = 0$, the normal velocity of the fluid at the surface, must be equal to the normal velocity of the surface; hence the sheet of fluid of which the boundary is composed, must always consist of the same elements of fluid. Hence

$$F(x + u\delta t,\ y + v\delta t,\ z + w\delta t,\ t + \delta t) = 0,$$

and therefore
$$\frac{dF}{dt} + u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz} = 0 \quad \text{..............(17).}$$

If the boundary is fixed, the condition becomes

$$lu + mv + nw = 0 \quad \text{......................(18),}$$

where l, m, n are the direction cosines of the normal to F.

Dynamical Theorems.

13. Before proceeding to discuss the equations of motion of a perfect fluid, it will be desirable to consider certain dynamical theorems.

The three fundamental propositions of Dynamics are the Principle of Linear Momentum, the Principle of Angular Momentum and the Principle of Energy. The first two propositions, which may be conveniently referred to as the Principle of Momentum, may be enunciated as follows:—

I. *The rate of change of the component of the linear momentum, parallel to an axis, of any dynamical system is equal to the component, parallel to that axis, of the impressed forces which act upon the system.*

II. *The rate of change of the component of the angular momentum about any axis, is equal to the moment of the impressed forces about that axis.*

The form in which the Principle of Energy most usually occurs in mechanical investigations is one which may be termed the *Principle of the Conservation of Mechanical Energy*, which asserts that:—

III. *If the system is not acted upon by any dissipative forces (such as internal friction which converts mechanical energy into heat), the sum of the kinetic and potential energies of the system is constant throughout the motion.*

Let ξ, λ be the components of the linear and angular momenta of the system along and about any *fixed* axis; X, L the components of the forces and couples which act upon the system. Then the Principle of Momentum is expressed by the two equations

$$d\xi/dt = X\,; \quad d\lambda/dt = L.$$

If the components of the force and couple are zero, we obtain by integration

$$\xi = \text{const.}, \quad \lambda = \text{const.},$$

which shows that the components of the linear and angular momenta are constant throughout the motion. These results are called the Principles of the *Conservation* of Linear and Angular Momentum.

14. For the proof of these theorems the reader is referred to treatises on Dynamics: but it may be worth while to show that the Principle of Linear Momentum is a direct consequence of Newton's Laws of Motion.

It is proved in treatises on Mechanics that the parallelogram of forces is a consequence of Newton's second law, and that the parallelogram of couples is a consequence of the parallelogram of forces; whence all the theorems relating to the composition and resolution of forces and couples are deductions from the second law.

Let m_1, m_2 be the masses of any two elements of a material system; (x_1, y_1, z_1), (x_2, y_2, z_2) their coordinates referred to *fixed* axes; let R_{12} be the molecular force of m_2 on m_1, and R_{21} of m_1 on m_2; also let F_1 be the force acting on m_1 which arises from causes which are independent of molecular action.

By Newton's second law, the forces R_{12} and F_1 may be resolved into components f_{12}, g_{12}, h_{12} and X_1, Y_1, Z_1 parallel to the axes; also by the same law the equations of motion of the elements are

$$d\,(m_1\dot{x}_1)/dt = X_1 + f_{12} + f_{13} + \ldots,$$
$$d\,(m_2\dot{x}_2)/dt = X_2 + f_{21} + f_{23} + \ldots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

By Newton's third law, the molecular force of m_2 on m_1 is equal in magnitude and opposite in direction to that of m_1 on m_2; whence $f_{12} = -f_{21}$, &c. Accordingly if we add the preceding system of equations, all the molecular forces disappear, and we obtain

$$\frac{d}{dt}\Sigma\,(m\dot{x}) = \Sigma X,$$

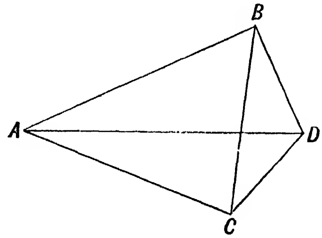
which is the analytical expression for the Principle of Linear Momentum. It should be noticed that we have not assumed that the molecular force between two elements of mass acts along the line joining them.

It should also be noticed that the Principle of Momentum is a proposition of a far more fundamental character than the Principle of the Conservation of Mechanical Energy; since the former proposition is true in the case of dissipative systems in which there is a conversion of mechanical energy into heat.

15. It has been already stated, that the pressure at every point of a perfect fluid is equal in all directions, whether the fluid

be at rest or in motion. It will now be shown that this property is the consequence of such a fluid being incapable of offering resistance to a tangential stress.

Let $ABCD$ be a small tetrahedron of fluid, and let p, p' be the pressures per unit of area upon the faces ABC and BCD. Also let ξ be the linear momentum of the element along a fixed axis parallel to AD, δv its volume, ρ its density, and X the force per unit of mass parallel to AD.



Since the projections of the faces ABC and BCD upon a plane perpendicular to AD are each equal to the same quantity δA, it follows that the equation of motion of the element is

$$\frac{d\xi}{dt} - \rho X \delta v - (p - p') \delta A = 0.$$

Now if $\dot{x}$ be the mean velocity parallel to AD of any point of the element, $\xi = \rho \dot{x} \delta v$ ultimately; accordingly when the element is indefinitely diminished the first two terms vanish in comparison with the third, and the equation reduces to $p = p'$, which proves the proposition.

The Equations of Motion.

16. The equations of motion of a perfect fluid may be obtained by two different methods, which we shall proceed to explain.

Let X, Y, Z be the components per unit of mass of the impressed forces (such as gravity and the like), which act upon the fluid; p the pressure, and ρ the density.

Let Q be any point of the fluid, whose coordinates are x, y, z; and consider an elementary parallelopiped δx, δy, δz (see figure to § 6) whose edges are parallel to the axes.

Let δm be the mass of an element of fluid contained within this parallelopiped, f_x the component of its acceleration parallel to x, and P_x the component parallel to x of the pressure due to the surrounding fluid upon the faces CB, AD.

Since the rate of change of the linear momentum parallel to

x of the fluid contained within the element is $\Sigma\,(f_x \delta m)$, it follows that the equation of motion is

$$\Sigma\,(f_x \delta m) = P_x + \Sigma(X\delta m) \qquad \text{.................(19).}$$

Now $$P_x = p\delta y\,\delta z - \left(p + \frac{dp}{dx}\,\delta x\right)\delta y\,\delta z,$$

whence (19) becomes

$$\Sigma\,(X - f_x)\,\delta m - \frac{dp}{dx}\,\delta x\,\delta y\,\delta z = 0 \qquad \text{.............(20).}$$

Since the parallelopiped is supposed to be indefinitely small, the first term of this equation becomes in the limit

$$(X - f_x)\,\Sigma\delta m = (X - f_x)\,\rho\,\delta x\,\delta y\,\delta z,$$

and therefore (20) becomes

$$\rho X - \frac{dp}{dx} = \rho f_x \qquad \text{......................(21).}$$

In all the applications which will occur, we shall use the flux method, in which case f_x is given by (3); whence (21) may be written

$$X - \frac{1}{\rho}\frac{dp}{dx} = \frac{\partial u}{\partial t}.$$

Resolving parallel to y and z and proceeding in a similar manner, we shall obtain two other equations, which may be deduced by cyclical interchange of the letters x, y, z, and u, v, w respectively; whence the equations of motion are

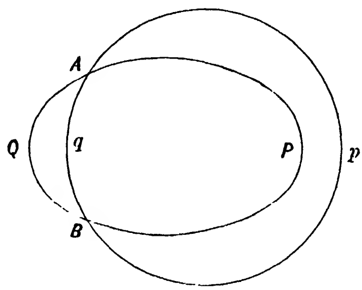
$$\left.\begin{aligned} X - \frac{1}{\rho}\frac{dp}{dx} &= \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} \\ Y - \frac{1}{\rho}\frac{dp}{dy} &= \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz} \\ Z - \frac{1}{\rho}\frac{dp}{dz} &= \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} \end{aligned}\right\} \qquad \text{..........(22).}$$

It thus appears that we can obtain the equations of motion of a perfect fluid solely by means of Newton's laws of motion, and without the aid of such adventitious devices as D'Alembert's Principle.

17. We shall now obtain the equations of motion by a different method, which will enable us to express the Principle of Linear Momentum in another form.

Let $APBQ$ represent the fluid, which at time t, is contained

within any imaginary closed surface S, described in the fluid. At the end of an interval δt, the fluid will no longer be contained within S, but will occupy a different position, which is shown by the line $ApBq$ in the figure.



Let M_x, $M_x + \delta M_x$, be the component momenta parallel to x, of the original fluid at times t, $t + \delta t$; X' and P_x the components parallel to x of the impressed forces, and the pressure upon the boundary of the given mass of fluid due to the action of the surrounding fluid.

By the Principle of Linear Momentum we obtain immediately the equation

$$\frac{dM_x}{dt} = X' + P_x \quad\text{.......................(23).}$$

Let $\delta\mu_1$, $M_x + \delta M'_x$, $\delta\mu_2$ be the component momenta parallel to x, of the fluid which at time $t + \delta t$ occupies the spaces $ApBPA$, $APBQA$ and $AqBQA$; then

$$M_x + \delta M_x = M_x + \delta M'_x + \delta\mu_1 - \delta\mu_2,$$

whence
$$\delta M_x = \delta M'_x + \delta\mu_1 - \delta\mu_2,$$

and therefore (23) may be written

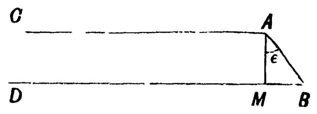
$$\frac{dM'_x}{dt} = \frac{d\mu_2}{dt} - \frac{d\mu_1}{dt} + X' + P_x \quad\text{.................(24).}$$

Now dM'_x/dt is the rate of increase of the component of momentum parallel to x, of the fluid contained within S; and $d\mu_2/dt - d\mu_1/dt$ is the rate at which momentum parallel to x, flows into S. Whence (24) asserts that,—

The rate of increase of the component of momentum parallel to x, of the fluid contained within any given closed surface S, is equal to the rate at which momentum parallel to x flows into S across the boundary of S, together with the rate at which momentum parallel to x, is generated by the component of the impressed force parallel

to x, and by the component parallel to x, of the pressure exerted by the surrounding fluid upon the boundary of S.

18. In order to apply this proposition, we must calculate the momentum which flows into a surface across its boundary.



Let AB be any element dS of the surface, CA the direction of motion of the fluid, q its resultant velocity, and ϵ the angle which its direction makes with the normal to dS drawn outwards. Through the perimeter of dS, describe a small cylinder, whose curved side contains the lines of flow which pass through the perimeter of dS; then if AM be the projection of dS upon a plane perpendicular to the lines of flow, $BAM = \epsilon$. Hence the total momentum which flows into S across the element dS in time δt is

$$\rho q dS \cos \epsilon \,.\, q\delta t = \rho q q' dS \delta t;$$

where $q' = q \cos \epsilon$, is the component velocity perpendicular to dS.

The component in any assigned direction of the momentum which flows into S, is found by multiplying this quantity by the cosine of the angle between this direction and the direction of q; and may therefore be written $\rho q'' q' dS \delta t$, where q'' is the velocity in the given direction.

Applying this to an elementary parallelopiped δx, δy, δz, we see at once that

$$\frac{d\mu_2}{dt} - \frac{d\mu_1}{dt} = -\left\{\frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz}\right\} \delta x \, \delta y \, \delta z.$$

Also
$$\frac{dM'_x}{dt} = \frac{d(\rho u)}{dt} \delta x \, \delta y \, \delta z,$$

$$X' = \rho X \delta x \, \delta y \, \delta z,$$

whence the equation for motion parallel to x is

$$\rho X - \frac{dp}{dx} = \frac{d(\rho u)}{dt} + \frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz}.$$

Performing the differentiations on the right-hand side, and taking account of the equation of continuity, this equation reduces to the first of (22).

19. The equations of motion together with the equation of continuity, furnish four relations between the five unknown quantities u, v, w, p, ρ; and are therefore not sufficient to determine the motion.

If however the fluid be a liquid, ρ is constant, and the above-mentioned equations together with the boundary conditions are sufficient to determine the motion; but in the case of a gas another equation is required, which is furnished by means of a relation which exists between p and ρ.

When the motion of the gas is such that the temperature remains constant, we have by Boyle's Law the equation

$$p = k\rho \quad\quad (25),$$

where k is a constant.

But when the motion is such as to cause a sudden compression or dilatation, an increase or decrease of temperature will be produced; and if it is assumed (as is the case with sound waves), that the compression is so sudden that loss or gain of heat by radiation may be neglected, it will be shown in the second part, that the required relation is

$$p = k'\rho^\gamma \quad\quad (26),$$

where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. For all gases this quantity has the approximately constant value 1·408.

20. Let us now suppose that the forces arise from a conservative system whose potential is V. Since p is a function of ρ, we may put

$$Q = -\int \frac{dp}{\rho} - V,$$

and the left-hand sides of (22) will be respectively equal to dQ/dx, dQ/dy, dQ/dz. If therefore we eliminate Q by differentiating the second equation with respect to z and the third with respect to y, and introduce the values of ξ, η, ζ from § 8, we shall obtain

$$\frac{\partial \xi}{\partial t} = \xi \frac{du}{dx} + \eta \frac{dv}{dx} + \zeta \frac{dw}{dx} - \xi\theta,$$

where ξ, η, ζ are the components of molecular rotation and $\theta = du/dx + dv/dy + dw/dz$. Eliminating θ by means of the equation of continuity $\partial\rho/\partial t + \rho\theta = 0$, and taking account of the

two other equations which may be written down from symmetry, we shall obtain

$$\left.\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right)&=\frac{\xi}{\rho}\frac{du}{dx}+\frac{\eta}{\rho}\frac{dv}{dx}+\frac{\zeta}{\rho}\frac{dw}{dx}\\ \frac{\partial}{\partial t}\left(\frac{\eta}{\rho}\right)&=\frac{\xi}{\rho}\frac{du}{dy}+\frac{\eta}{\rho}\frac{dv}{dy}+\frac{\zeta}{\rho}\frac{dw}{dy}\\ \frac{\partial}{\partial t}\left(\frac{\zeta}{\rho}\right)&=\frac{\xi}{\rho}\frac{du}{dz}+\frac{\eta}{\rho}\frac{dv}{dz}+\frac{\zeta}{\rho}\frac{dw}{dz}\end{aligned}\right\}\dots\dots\dots\dots(27).$$

These equations may also be written in the form

$$\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right)=\frac{\xi}{\rho}\frac{du}{dx}+\frac{\eta}{\rho}\frac{du}{dy}+\frac{\zeta}{\rho}\frac{du}{dz}, \text{ \&c. \&c.}$$

21. It was stated in § 7, that in many important problems, the motion is such that a velocity potential exists. The condition that such should be the case is, that ξ, η, ζ should each vanish. We shall now prove that, when the fluid is under the action of a conservative system of forces, a velocity potential will always exist whenever it exists at any particular instant.

Let us choose the particular instant at which a velocity potential exists, as the origin of the time; then by hypothesis ξ, η, ζ vanish when $t=0$; also the coefficients of these quantities in (27), will not become infinite at any point of the interior of the fluid; it will therefore be possible to determine a quantity L, which shall be a superior limit to the numerical values of these coefficients. Hence ξ, η, ζ cannot increase faster than if they satisfied the equation

$$\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right)=\frac{L}{\rho}(\xi+\eta+\zeta), \text{ \&c. \&c.}$$

But if $\xi+\eta+\zeta=\Omega\rho$, we obtain by adding the above equations

$$\frac{\partial\Omega}{\partial t}=3L\Omega,$$

whence $$\Omega=Ae^{3Lt}.$$

Now $\Omega=0$ when $t=0$, therefore $A=0$; and since Ω is the sum of three quantities each of which is essentially positive, it follows that ξ, η, ζ must always remain zero, if they are so at any particular instant. The above proof is due to Sir G. Stokes[1].

[1] "On the friction of fluids in motion," Section II. *Trans. Camb. Phil. Soc.* vol. VIII.

22. There is, as was first shown by Sir G. Stokes, an important physical distinction in the character of the motion which takes place, according as a velocity potential does or does not exist.

Conceive an indefinitely small spherical element of a fluid in motion to become suddenly solidified, and the fluid about it to be suddenly destroyed. By the instantaneous solidification, velocities will be suddenly generated or destroyed in the different portions of the element, and a set of mutual impulsive forces will be called into action.

Let x, y, z be the coordinates of the centre of inertia G of the element at the instant of solidification, $x+x', y+y', z+z'$ those of any other point P in it; let u, v, w be the velocities of G along the three axes just before solidification, u', v', w' the velocities of P relative to G; also let $\bar{u}, \bar{v}, \bar{w}$ be the velocities of G, u_1, v_1, w_1 the relative velocities of P, and ξ, η, ζ the angular velocities just after solidification. Since all the impulsive forces are internal, we have

$$\bar{u}=u, \quad \bar{v}=v, \quad \bar{w}=w.$$

We have also by the Principle of Conservation of Angular Momentum,

$$\Sigma m\{y'(w_1-w')-z'(v_1-v')\}=0, \text{ \&c.}$$

m denoting an element of the mass of the element considered.

But $u_1=\eta z'-\zeta y'$, and u' is ultimately equal to

$$\frac{du}{dx}x'+\frac{du}{dy}y'+\frac{du}{dz}z',$$

and similar expressions hold good for the other quantities. Substituting in the above equation, and observing that

$$\Sigma my'z'=\Sigma mz'x'=\Sigma mx'y'=0, \text{ and } \Sigma mx'^2=\Sigma my'^2=\Sigma mz'^2,$$

we have $$\xi=\tfrac{1}{2}\left(\frac{dw}{dy}-\frac{dv}{dz}\right), \text{ \&c.}$$

We see then that an indefinitely small spherical element of the fluid, if suddenly solidified and detached from the rest of the fluid, will begin to move with a motion of translation alone, or a motion of translation combined with one of rotation, according as $udx+vdy+wdz$ is, or is not, an exact differential, and in the latter case the angular velocities will be determined by the equations

$$2\xi=\frac{dw}{dy}-\frac{dv}{dz}, \quad 2\eta=\frac{du}{dz}-\frac{dw}{dx}; \quad 2\zeta=\frac{dv}{dx}-\frac{du}{dy}.$$

On account of the physical meaning of the quantities ξ, η, ζ, they are called the *components of molecular rotation*, and motion which is such that they do not vanish is called *rotational or vortex motion;* when they vanish, the motion is called *irrotational.*

In the foregoing investigation, it has been assumed that the pressure is a function of the density, and also that the fluid is under the action of a conservative system of forces; it therefore follows that vortex motion cannot be produced and, if once set up, cannot be destroyed by such a system of forces. It can however be shown that the theorem is not true if the pressure is not a function of the density. If therefore by reason of any chemical action, the pressure should cease to be a function of the density during any interval of time however short, vortex motion might be produced, or if in existence might be destroyed.

23. The equations of motion can be integrated whenever a force and a velocity potential exist; for putting

$$Q = -\int \frac{dp}{\rho} - V,$$

and multiplying (22) by dx, dy, dz respectively and adding, we obtain

$$dQ = \frac{\partial u}{\partial t}\,dx + \frac{\partial v}{\partial t}\,dy + \frac{\partial w}{\partial t}\,dz.$$

Now in the present case $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y} + \frac{w\partial u}{\partial z}$

$$\frac{\partial u}{\partial t} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{dv}{dx} + w\frac{dw}{dx}$$

$$= \frac{d}{dx}\left(\frac{d\phi}{dt} + \tfrac{1}{2}q^2\right),$$

where q is the resultant velocity. Integrating, we obtain

$$\int \frac{dp}{\rho} + V + \frac{d\phi}{dt} + \tfrac{1}{2}q^2 = F(t) \quad\text{.................}(28),$$

where F is an arbitrary function.

24. When the motion of a liquid is steady, du/dt, dv/dt and dw/dt are each zero, and in this case the general equations of motion can always be integrated. It will however be necessary to distinguish between irrotational and rotational motion.

In the former case $d\phi/dt = 0$ and $F(t)$ is constant; whence a first integral is

$$p/\rho + V + \tfrac{1}{2}q^2 = C \qquad (29).$$

This result, from the name of its discoverer, is called Bernoulli's Theorem.

When the motion is rotational, let

$$R = p/\rho + V + \tfrac{1}{2}q^2,$$

and the general equations of motion may be written

$$\left.\begin{aligned} \frac{du}{dt} - 2v\zeta + 2w\eta &= -\frac{dR}{dx} \\ \frac{dv}{dt} - 2w\xi + 2u\zeta &= -\frac{dR}{dy} \\ \frac{dw}{dt} - 2u\eta + 2v\xi &= -\frac{dR}{dz} \end{aligned}\right\} \qquad (30).$$

(i) Let the motion be steady and in two dimensions; then $w = \xi = \eta = 0$, and none of the quantities are functions of z or t; whence substituting the values of u and v from (14) in terms of ψ we obtain

$$\frac{dR}{dx} = -2\zeta\frac{d\psi}{dx}, \quad \frac{dR}{dy} = -2\zeta\frac{d\psi}{dy},$$

which shows that $2\zeta = F'(\psi)$, where F is an arbitrary function; and therefore

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + F(\psi) = 0 \qquad (31).$$

From these results it follows that the sum of the first three terms is constant along a stream line, but varies as we pass from one stream line to another; also that the molecular rotation is constant along a stream line.

(ii) In the same way it can be shown that when the motion is symmetrical with respect to an axis,

$$2\omega/\varpi = F'(\psi),$$

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + F(\psi) = 0 \qquad (32),$$

where ω is the molecular rotation, and ψ is Stokes' current function.

(iii) In the general case of the steady motion of a liquid, multiply (30) by u, v, w and add and we obtain

$$u\frac{dR}{dx} + v\frac{dR}{dy} + w\frac{dR}{dz} = 0 \qquad (33).$$

Multiply by ξ, η, ζ and add, and we obtain

$$\xi \frac{dR}{dx} + \eta \frac{dR}{dy} + \zeta \frac{dR}{dz} = 0 \quad\quad (34).$$

These equations show that R is constant along a stream line and a vortex line; whence it is possible to draw a family of surfaces each of which is covered with a network of stream lines and vortex lines. Let $\lambda =$ const. be such a surface; then since λ contains stream lines and vortex lines, it follows that

$$u\lambda_x + v\lambda_y + w\lambda_z = 0,$$
$$\xi\lambda_x + \eta\lambda_y + \zeta\lambda_z = 0,$$

where $\lambda_x = d\lambda/dx$, &c. These equations show that if we write $R = -F(\lambda)$ equations (33) and (34) will be satisfied; whence a first integral of the equations of motion is

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + F(\lambda) = 0 \quad\quad (35),$$

where λ is a surface which contains both stream lines and vortex lines.

Impulsive Motion.

25. The equations which determine the change of motion when a fluid is acted upon by impulsive forces, may be deduced in manner similar to that employed in § 16.

Let u, v, w and u', v', w' be the velocities of the fluid just before and just after the impulse; p the impulsive pressure.

Since impulsive forces are equal to the change of momentum which they produce, it follows by considering the motion of a small parallelopiped $\delta x \delta y \delta z$, that

$$\rho(u' - u)\,\delta x \delta y \delta z = p\,\delta y \delta z - \left(p + \frac{dp}{dx}\,\delta x\right)\delta y \delta z,$$

whence the equations of impulsive motion are

$$\left.\begin{aligned} \rho(u' - u) &= -\frac{dp}{dx} \\ \rho(v' - v) &= -\frac{dp}{dy} \\ \rho(w' - w) &= -\frac{dp}{dz} \end{aligned}\right\} \quad\quad (36).$$

Multiplying by dx, dy, dz and adding we obtain

$$-dp/\rho=(u'-u)\,dx+(v'-v)\,dy+(w'-w)\,dz \qquad (37).$$

In the case of a liquid ρ is constant, whence differentiating with respect to x, y, z, adding and taking account of the equation of continuity, we obtain

$$\nabla^2 p=0 \qquad (38).$$

If the liquid were originally at rest, it is clear that the motion produced by the impulse must be irrotational, whence if ϕ be its velocity potential

$$p=-\rho\phi \qquad (39),$$

which is a very important result.

Flow and Circulation.

26. The line integral $\int(udx+vdy+wdz)$, taken along any curve joining a fixed point A with a variable point P, is called the *flow* from A to P.

If the points A and P coincide, so that the curve along which the integration takes place is a closed curve, this line integral is called the *circulation* round the closed curve.

If the motion of a liquid is irrotational, and ϕ_A, ϕ_P denote the values of the velocity potential at A and P, the flow from A to P is simply $\phi_P-\phi_A$, and is independent of the path from A to P; also the circulation round any closed curve is zero, *provided ϕ be a single-valued function.* Cases however occur in which ϕ is a *many-valued function;* and when this is the case, the value of the circulation will depend upon the position of the closed curve round which the integration is taken, being zero for some curves, whilst for others it has a finite value.

For example, when the motion is in two dimensions, ϕ satisfies the equation

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0,$$

and it can be verified by trial, that a particular solution of this equation is

$$\phi=m\tan^{-1}y/x.$$

This value of ϕ therefore gives a possible kind of irrotational motion. Let θ be the *least* value of the angle $\tan^{-1}y/x$; then since

the equation $\theta = \tan^{-1} y/x$ is satisfied by $\theta + 2n\pi$, where n is any positive or negative integer, it follows that the most general value of ϕ is

$$\phi = m\theta + 2mn\pi,$$

whence ϕ is a many-valued function.

Let a point P start from any position, and describe a closed curve which does not surround the origin. During the passage of P from its original to its final position, the angle θ increases to a certain value, then diminishes, and finally arrives at its original value, and therefore the circulation round such a curve is zero; but if the closed curve surrounds the origin, θ increases from its original value to $2\pi + \theta$, as the point travels round the closed curve, and therefore the circulation round a curve which encloses the origin is $2m\pi$.

Irrotational motion which is characterized by a single-valued velocity potential, is called *acyclic irrotational motion;* whilst motion which is characterized by a many-valued velocity potential, is called *cyclic irrotational motion.*

27. The importance of the distinction between cyclic and acyclic motion will not be fully understood, until we discuss the theory of rectilinear vortex motion; but the results of § 25 will enable us to prove, that cyclic motion cannot be produced or destroyed by impulsive forces.

Integrate (37) round any closed curve, then since p/ρ (or $\int\rho^{-1}dp$ in the case of a gas) is necessarily a single-valued function, it vanishes when integrated round any closed curve, and we obtain

$$\int(u'dx + v'dy + w'dz) = \int(udx + vdy + wdz),$$

which shows that the circulation is unaltered by the impulse.

We can also show that cyclic irrotational motion cannot be generated nor destroyed, when the liquid is under the action of forces having a single-valued potential; for if we put

$$H = \int\frac{dp}{\rho} + V + \frac{d\phi}{dt} + \tfrac{1}{2}q^2,$$

the equations of motion are

$$\frac{dH}{dx} = 0,\ \frac{dH}{dy} = 0,\ \frac{dH}{dz} = 0.$$

Multiply these equations by dx, dy, dz, add and integrate round a closed curve, and let κ be the circulation; we obtain

$$\int\frac{dp}{\rho}+(V+\tfrac{1}{2}q^2)_2-(V+\tfrac{1}{2}q^2)_1+\frac{d\phi}{dt}+\frac{d\kappa}{dt}-\frac{d\phi}{dt}=0 \quad\ldots(40),$$

where the suffixes refer to the initial and final positions of the moving point. Since $\int\rho^{-1}dp$ and $V+\frac{1}{2}q^2$ are single-valued functions, the sum of the first three terms is zero, and (40) reduces to

$$\frac{d\kappa}{dt}=0,$$

whence $$\kappa = \text{const.}$$

If therefore κ is zero, or the motion is acyclic, it will remain zero during the subsequent motion.

Sources, Doublets and Images.

28. When the motion of a liquid is irrotational and symmetrical with respect to a fixed point, which we shall choose as the origin, the value of ϕ at any other point P is a function of the distance alone of P from the origin; and Laplace's equation becomes

$$\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}=0.$$

Therefore $$\phi=-\frac{m}{r},$$

and $$\frac{d\phi}{dr}=\frac{m}{r^2}.$$

The origin is therefore a singular point, from or to which the stream lines diverge or converge, according as m is positive or negative. In the former case the singular point is called a *source*, in the latter case a *sink*.

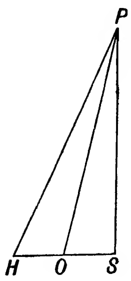
The flux across any closed surface surrounding the origin is,

$$\iint\frac{d\phi}{dr}\,dS=\iint\frac{m\cos\epsilon}{r^2}\,dS=m\iint d\Omega$$
$$=4\pi m,$$

where $d\Omega$ is the solid angle subtended by dS at the origin, and ϵ is the angle which the direction of motion makes with the normal to S drawn outwards.

The constant m is called the strength of the source.

29. A *doublet* is formed by the coalescence of an equal source and sink. To find its velocity potential, let there be a source and sink at S and H respectively, and let O be the middle point of SH, then



$$\phi = -\frac{m}{SP} + \frac{m}{HP}$$

$$= -\frac{mSH\cos SOP}{OP^2}$$

Now let SH diminish and m increase indefinitely, but so that the product $m . SH$ remains finite and equal to μ, then

$$\phi = -\frac{\mu\cos SOP}{r^2}$$

$$= -\frac{\mu z}{r^3},$$

if the axis of z coincides with OS.

Hence the velocity potential due to a doublet, is equal to the magnetic potential of a small magnet whose axis coincides with the axis of the doublet, and whose negative pole corresponds to the source end of the doublet.

30. When the motion is in two dimensions, and is symmetrical with respect to the axis of z, Laplace's equation becomes

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0.$$

Therefore $$\phi = m\log r,$$

$$\frac{d\phi}{dr} = \frac{m}{r},$$

where r is the distance of any point from the axis. This value of ϕ represents a line source of infinite length, whose strength per unit of length is equal to m.

If ψ be the current function,

$$\frac{m}{r} = \frac{1}{r}\frac{d\psi}{d\theta}.$$

Therefore $$\psi = m\theta$$

$$= m\tan^{-1}\frac{y}{x}.$$

31. The velocity potential due to a doublet in two-dimensional motion is

$$\begin{aligned}\phi &= m \log SP - m \log HP \\ &= -m \frac{SH}{OP} \cos SOP = -\frac{\mu \cos SOP}{r} \\ &= -\frac{\mu x}{r^2}.\end{aligned}$$

Theory of Images.

32. Let H_1, H_2 be any two hydrodynamical systems situated in an infinite liquid. Since the lines of flow either form closed curves or have their extremities in the singular points or boundaries of the liquid, it will be possible to draw a surface S, which is not cut by any of the lines of flow, and over which there is therefore no flux, such that the two systems H_1, H_2 are completely shut off from one another.

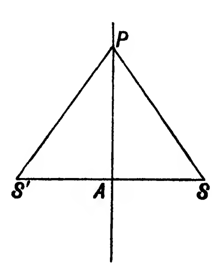
The surface S may be either a closed surface such as an ellipsoid, or an infinite surface such as a paraboloid.

If therefore we remove one of the systems (say H_2) and substitute for it such a surface as S, everything will remain unaltered on the side of S on which H_1 is situated; hence the velocity of the liquid due to the combined effect of H_1 and H_2 will be the same as the velocity due to the system H_1 in a liquid which is bounded by the surface S.

The system H_2 is called the *image* of H_1 with respect to the surface S, and is such that if H_2 were introduced and S removed, there would be no flux across S.

The method of images was invented by Lord Kelvin, and has been developed by Helmholtz, Maxwell and other writers; it affords a powerful method of solving many important physical problems.

33. We shall now give some examples.



Let S, S' be two sources whose strengths are m. Through A the middle point of SS' draw a plane at right angles to SS'. The normal component of the velocity of the liquid at any point P on this plane is

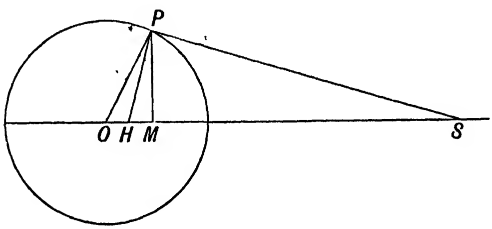
$$-\frac{m}{SP^2} \cos PSA + \frac{m}{S'P^2} \cos PS'A = 0.$$

Hence there is no flux across AP. If therefore Q be any point on the right-hand side of AP, the velocity potential due to a source at S, in a liquid which is bounded by the fixed plane AP, is

$$\phi = -\frac{m}{SQ} - \frac{m}{S'Q}.$$

Hence the image of a source S with respect to a plane is an equal source, situated at a point S' on the other side of the plane, whose distance from it is equal to that of S.

34. The image in a sphere, of a doublet whose axis passes through the centre of the sphere, can also be found by elementary methods.



Let S be the doublet, O the centre of the sphere, a its radius, and let $OS = f$.

The velocity potential of a doublet situated at the origin and whose axis coincides with OS, has already been shown to be

$$\phi = -\frac{m \cos\theta}{r^2};$$

whence if R, Θ be the radial and transversal velocities

$$R = \frac{d\phi}{dr} = \frac{2m\cos\theta}{r^3},$$

$$\Theta = \frac{1}{r}\frac{d\phi}{d\theta} = \frac{m\sin\theta}{r^3}.$$

Hence if we have a doublet at S, the component velocity along OP is

$$-\frac{2m}{SP^3}\cos OSP \cos OPS - \frac{m}{SP^3}\sin OSP \sin OPS$$

$$= -\frac{m}{SP^3}\{\cos OSP \cos OPS + \cos(OPS - OSP)\} \quad \text{......(41).}$$

Let us take a point H inside the sphere such that $OH = a^2/f$;

then it is known from geometry that the triangles OPH and OSP are similar, and therefore the preceding expression may be written

$$-\frac{m}{SP^3}\{\cos OPH \cos OHP + \cos SPH\}.$$

But the normal velocity due to a doublet of strength m' placed at H is by (41)

$$-\frac{m'}{HP^3}\{\cos OPH \cos OHP + \cos SPH\},$$

and therefore the normal velocity will be zero if

$$\frac{m}{SP^3} + \frac{m'}{HP^3} = 0$$

for all positions of P. But by a well-known theorem,

$$\frac{f}{SP} = \frac{a}{HP},$$

and therefore the condition that the normal velocity should vanish, is that

$$m' = -ma^3/f^3.$$

Whence the image of a doublet of strength m in a liquid bounded by a sphere is another doublet placed at the inverse point H, whose strength is $-ma^3/f^3$.

The theory of sources, sinks and doublets furnishes a powerful method of solving certain problems relating to the motion of solid bodies in a liquid[1].

We shall conclude this chapter by working out some examples.

35. *A mass of liquid whose external surface is a sphere of radius* a, *and which is subject to a constant pressure* Π, *surrounds a solid sphere of radius* b. *The solid sphere is annihilated, it is required to determine the motion of the liquid.*

It is evident that the only possible motion which can take

[1] If a magnetic system be suddenly introduced into the neighbourhood of a conducting spherical shell, it can be shown that the effect of the induced currents at points outside the shell, is *initially* equivalent to a magnetic system inside the shell, which is the hydrodynamical image of the external system; and that the law of decay of the currents, is obtained by supposing the radius of the shell to diminish according to the law $ae^{-\sigma t/4\pi a}$, where σ is the specific resistance of the shell. Analogous results hold good in the case of a plane current sheet; hence all results concerning hydrodynamical images in spheres and planes are capable of an electromagnetic interpretation. See C. Niven, *Phil. Trans.* 1881.

place, is one in which each element of liquid moves towards the centre, whence the free surfaces will remain spherical. Let R', R be their external and internal radii at any subsequent time, r the distance of any point of the liquid from the centre. The equation of continuity is

$$\frac{d}{dr}(r^2 v) = 0,$$

whence
$$r^2 v = F(t).$$

The equation for the pressure is

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{dv}{dt} - v\frac{dv}{dr}$$

$$= -\frac{F'(t)}{r^2} - \tfrac{1}{2}\frac{dv^2}{dr}$$

whence
$$\frac{p}{\rho} = A + \frac{F'(t)}{r} - \tfrac{1}{2}v^2.$$

When $r = R'$, $p = \Pi$; and when $r = R$, $p = 0$; whence if V, V' be the velocities of the internal and external surfaces

$$\frac{\Pi}{\rho} = F'(t)\left(\frac{1}{R'} - \frac{1}{R}\right) - \tfrac{1}{2}(V'^2 - V^2).$$

Since the volume of the liquid is constant,

$$R'^3 - R^3 = a^3 - b^3 = c^3,$$

also
$$F'(t) = \frac{d}{dt}(R^2 V),$$

whence

$$\frac{\Pi}{\rho} = V\frac{d}{dR}(R^2 V)\left\{\frac{1}{(R^3 + c^3)^{\frac{1}{3}}} - \frac{1}{R}\right\} - \tfrac{1}{2}V^2\left\{\frac{R^4}{(R^3 + c^3)^{\frac{4}{3}}} - 1\right\}.$$

Putting $z = R^4 V^2$. multiplying by $2R^2$ and integrating, we obtain

$$\tfrac{2}{3}\frac{\Pi(R^3 - b^3)}{\rho R^4} = V^2\left\{\frac{1}{(R^3 + c^3)^{\frac{1}{3}}} - \frac{1}{R}\right\},$$

which determines the velocity of the inner surface.

If the liquid had extended to infinity, we must put $c = \infty$, and we obtain

$$\frac{2\Pi}{3\rho}(b^3 - R^3) = R^3\left(\frac{dR}{dt}\right)^2,$$

whence if t be the time of filling up the cavity

$$t = \sqrt{\frac{3\rho}{2\Pi}} \int_0^b \frac{R^{\frac{3}{2}}\, dR}{\sqrt{b^3 - R^3}}.$$

Putting $b^3 x = R^3$, this becomes

$$t = b\sqrt{\frac{\rho}{6\Pi}} \int_0^1 x^{-\frac{1}{6}} (1-x)^{-\frac{1}{2}} dx$$

$$= b\sqrt{\frac{\pi\rho}{6\Pi}} \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})}.$$

The preceding example may be solved at once by the Principle of Energy.

The kinetic energy of the liquid is

$$2\pi\rho \int_R^{R'} r^2 v^2 dr = 2\pi\rho V^2 R^4 \int_R^{R'} \frac{dr}{r^2}$$

$$= 2\pi\rho V^2 R^4 \left\{ \frac{1}{R} - \frac{1}{(R^3 + c^3)^{\frac{1}{3}}} \right\}.$$

The work done by the external pressure is

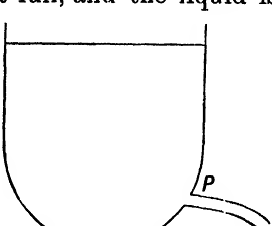
$$4\pi\Pi \int_R^a r^2 dr = \tfrac{4}{3}\Pi\pi (a^3 - R'^3)$$

$$= \tfrac{4}{3}\Pi\pi (b^3 - R^3),$$

whence
$$\tfrac{2}{3}\Pi (b^3 - R^3) = V^2 R^4 \rho \left\{ \frac{1}{R} - \frac{1}{(R^3 + c^3)^{\frac{1}{3}}} \right\}.$$

36. The determination of the motion of a liquid in a vessel of any given shape is one of great difficulty, and the solution has been effected in only a comparatively few number of cases. If, however, liquid is allowed to flow out of a vessel, the inclinations of whose sides to the vertical are small, an approximate solution may be obtained by neglecting the horizontal velocity of the liquid. This method of dealing with the problem is called the hypothesis of parallel sections.

Let us suppose that the vessel is kept full, and the liquid is allowed to escape by a small orifice at P. Let h be the distance of P below the free surface, and z that of any element of liquid. Since the motion is steady, the equation for the pressure will be



$$\frac{p}{\rho} - gz + \tfrac{1}{2}v^2 = C.$$

Now if the orifice be small in comparison with the area of the top of the vessel, the velocity at the free surface will be so small that it may be neglected; hence if Π be the atmospheric pressure, when $z=0$, $p=\Pi$, $v=0$ and therefore $C=\Pi/\rho$. At the orifice $p=\Pi$, $z=h$, whence the velocity of efflux is

$$v=\sqrt{(2gh)},$$

and is therefore the same as that acquired by a body falling from rest, through a height equal to the depth of the orifice below the upper surface of the liquid. This result is called *Torricelli's Theorem.*

The Vena Contracta.

37. When a jet of liquid escapes from a small hole in the bottom of a cistern, it is found that the area of the jet is less than the area of the hole; so that if σ be the area of the hole and σ' that of the jet, the ratio σ'/σ, which is called the *coefficient of contraction* of the jet, is always less than unity. We shall now show that this ratio must always be greater than $\frac{1}{2}$.

We shall suppose for simplicity, that no forces are in action, and that the jet escapes in vacuo; we shall also suppose that the upper surface of the liquid is subjected to a pressure p.

If the hole were absent, the pressure would be equal to p throughout the vessel, and therefore since the hole is small, the pressure may be taken to be sensibly equal to p except just in the neighbourhood of the hole, where it is zero.

If σ'' be the area of the cistern, v'' the velocity of the liquid across any section which is at some distance from the hole, the momentum which flows in across this section per unit of time is $\rho\sigma''v''^2$ and the momentum which flows out of the hole is $\rho\sigma'v'^2$ whence by the principle stated at the end of § 17

$$\rho\sigma''v''^2-\rho\sigma'v'^2=p\sigma''-p(\sigma''-\sigma)=p\sigma.$$

But since the pressure is zero at the hole

$$p/\rho-\tfrac{1}{2}v''^2=-\tfrac{1}{2}v'^2.$$

Also the equation of continuity is

$$\sigma''v''=\sigma'v',$$

whence eliminating p, v', v'' we obtain

$$\frac{2}{\sigma}=\frac{1}{\sigma'}+\frac{1}{\sigma''},$$

which shows that the coefficient of contraction is greater than $\frac{1}{2}$.

The quantity of liquid which flows out of the vessel per unit of time is therefore $\rho\sigma' v'$. Now if σ is small compared with σ'', we may neglect σ''^{-1}, and therefore $\sigma'=\frac{1}{2}\sigma$; hence the discharge is equal to

$$\tfrac{1}{2}\rho\sigma v',$$

where v' is the velocity of efflux.

Giffard's Injector[1].

38. If we suppose fluid of density ρ to escape through a small hole, from a large closed vessel in which the pressure is p at points where the motion is insensible, into an open space in which the pressure is Π, then if q be the velocity of efflux,

$$\Pi+\tfrac{1}{2}\rho q^2=C,\quad p=C;$$

whence
$$q=\sqrt{\{2(p-\Pi)/\rho\}}.$$

If A be the sectional area of the jet at the vena contracta, the quantity of fluid which escapes per unit of time is

$$\tfrac{1}{2}A\rho q=A\ \{2\rho(p-\Pi)\}.$$

The momentum per unit of time is

$$A\rho q^2=2A(p-\Pi).$$

The energy per unit of time is

$$\tfrac{1}{2}A\rho q^3=A(p-\Pi)^{\frac{3}{2}}\sqrt{(2/\rho)}.$$

In Giffard's Injector, a jet of steam issuing by a pipe from the upper part of the boiler, is directed at an equal pipe leading back into the lower part of the boiler, the jet being kept constantly just surrounded with water. Now if we assume that the velocity of the steam jet is equal to the velocity at which the water flows into the pipe leading to the lower part of the boiler, which must be very nearly true; it follows from the preceding equations that

$$\frac{\text{velocity of steam jet}}{\text{velocity of water jet}}=\sqrt{\frac{\rho}{\sigma}},$$

$$\frac{\text{quantity of steam jet}}{\text{quantity of water jet}}=\sqrt{\frac{\sigma}{\rho}},$$

[1] Greenhill, Art. Hydromechanics, *Encyc. Brit.*

$$\frac{\text{momentum of steam jet}}{\text{momentum of water jet}} = 1,$$

$$\frac{\text{energy of steam jet}}{\text{energy of water jet}} = \sqrt{\frac{\rho}{\sigma}},$$

where σ is the density of the steam jet.

If the steam and water jets were directed at each other with a small interval between them, the superior energy and equal momentum of the steam jet would overcome the water jet, and steam would be driven back into the boiler. But the steam jet without losing its momentum, is capable of being mixed with water to such an extent, as to become a condensed water jet moving with the velocity of the water jet, and still entering the boiler, a valve preventing the reversal of the motion. Consequently the amount of water carried into the boiler per unit of time, will theoretically at most be the difference between the quantities which would escape by the water and steam jets, and therefore

$$= A\,(p - \Pi)^{\frac{1}{2}}\,(\sqrt{2\rho} - \sqrt{2\sigma});$$

and therefore the efficiency of the jet, i.e. the ratio of the quantity of water pumped in, to the quantity of steam used, will be

$$\sqrt{\frac{\rho}{\sigma}} - 1.$$

EXAMPLES.

1. Find the equation of continuity in a form suitable for air in a tube, and prove that if the density be $f(at - x)$ where t is the time and x the distance from one end of a uniform tube, the velocity is

$$\frac{af(at - x) + (V - a)f(at)}{f(at - x)},$$

where V is the velocity at that end of the tube.

2. If the motion of a liquid be in two dimensions, prove that if at any instant the velocity be everywhere the same in magnitude, it is so in direction.

3. If every particle of a fluid move on the surface of a sphere, prove that the equation of continuity is

$$\frac{d\rho}{dt}\cos\theta + \frac{d}{d\theta}(\rho\omega\cos\theta) + \frac{d}{d\phi}(\rho\omega'\cos\theta) = 0,$$

where ρ is the density, θ and ϕ the latitude and longitude of any element, and ω, ω' the angular velocities of the element in latitude and longitude respectively.

4. Fluid is moving in a fine tube of variable section κ, prove that the equation of continuity is

$$\frac{d}{dt}(\kappa\rho) + \frac{d}{ds}(\kappa\rho v) = 0,$$

where v is the velocity at the point s.

5. If $F(x, y, z, t)$ is the equation of a moving surface, the velocity of the surface normal to itself is

$$-\frac{1}{R}\frac{dF}{dt} \text{ where } R^2 = (dF/dx)^2 + (dF/dy)^2 + (dF/dz)^2.$$

6. If x, y and z are given functions of a, b, c and t, where a, b and c are constants for any particular element of fluid, and if u, v and w are the values of $\dot{x}$, $\dot{y}$, $\dot{z}$ when a, b, c are eliminated, prove analytically that

$$\frac{d^2x}{dt^2} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}.$$

7. If the lines of flow of a fluid lie on the surfaces of coaxial cones having the same vertex, prove that the equation of continuity is

$$r\frac{d\rho}{dt} + r\frac{d}{dr}(u\rho) + 2\rho u + \operatorname{cosec}\theta\frac{d}{d\phi}(\rho v) = 0.$$

8. Show that

$$x^2/(akt^2)^2 + kt^2\{(y/b)^2 + (z/c)^2\} = 1$$

is a possible form of the bounding surface at time t of a liquid.

9. A fine tube whose section k is a function of its length s, in the form of a closed plane curve of area A filled with ice, is moved in any manner. When the component angular velocity of the tube about a normal to its plane is Ω, the ice melts without change of volume. Prove that the velocity of the liquid relatively

to the tube at a point where the section is K, at any subsequent time when ω is the angular velocity is

$$\frac{2Ac}{K}(\Omega - \omega),$$

where $1/c = \int k^{-1}ds$, the integral being taken once round the tube.

10. A centre of force attracting inversely as the square of the distance, is at the centre of a spherical cavity within an infinite mass of liquid, the pressure on which at an infinite distance is ϖ, and is such that the work done by this pressure on a unit of area through a unit of length, is one half the work done by the attractive force on a unit of volume of the liquid from infinity to the initial boundary of the cavity; prove that the time of filling up the cavity will be

$$\pi a \sqrt{\frac{\rho}{\varpi}} \left\{2 - \left(\frac{3}{2}\right)^{\frac{3}{2}}\right\},$$

a being the initial radius of the cavity, and ρ the density of the liquid.

11. A solid sphere of radius a is surrounded by a mass of liquid whose volume is $4\pi c^3/3$, and its centre is a centre of attractive force varying directly as the square of the distance. If the solid sphere be suddenly annihilated, show that the velocity of the inner surface when its radius is x, is given by

$$\dot{x}^2 x^3 \{(x^3 + c^3)^{\frac{1}{3}} - x\} = \left(\frac{2\Pi}{3\rho} + \frac{2}{9}\mu c^3\right)(a^3 - x^3)(c^3 + x^3)$$

where ρ is the density, Π the external pressure and μ the absolute force.

12. Prove that if ϖ be the impulsive pressure, ϕ, ϕ' the velocity potentials immediately before and after an impulse acts, V the potential of the impulses,

$$\varpi + \rho V + \rho(\phi' - \phi) = \text{const.}$$

13. If the motion of a homogeneous liquid be given by a single valued velocity potential, prove that the angular momentum of any spherical portion of the liquid about its centre is always zero.

14. Homogeneous liquid is moving so that

$$u = \gamma x + \alpha y, \quad v = \beta x - \gamma y, \quad w = 0,$$

and a long cylindrical portion whose section is small, and whose axis is parallel to the axis of z, is solidified and the rest of the liquid destroyed. Prove that the initial angular velocity of the cylinder is

$$\frac{B\beta - A\alpha - 2F\gamma}{A+B},$$

where A, B, F are the moments and products of inertia of the section of the cylinder about the axes.

15. Fluid is contained within a sphere of small radius; prove that the momentum of the mass in the direction of the axis of x is greater than it would be if the whole were moving with the velocity at the centre by

$$\frac{Ma^2}{5\rho}\left\{\rho_x u_x + \rho_y u_y + \rho_z u_z + \tfrac{1}{2}\rho\nabla^2 u\right\},$$

where $\rho_x = d\rho/dx$ &c.

16. The motion of a liquid is in two dimensions, and there is a constant source at one point A in the liquid and an equal sink at another point B; find the form of the stream lines, and prove that the velocity at a point P varies as $(AP \,.\, BP)^{-1}$, the plane of the motion being unlimited.

If the liquid is bounded by the planes $x=0$, $x=a$, $y=0$, $y=a$, and if the source is at the point $(0, a)$ and the sink at $(a, 0)$, find an expression for the velocity potential.

17. The boundary of a liquid consists of an infinite plane having a hemispherical boss, whose radius is a and centre O. A doublet of unit strength is situated at a point S, whose axis coincides with OS, where OS is perpendicular to the plane. P is any point on the plane, $OP = y$, $OS = f$. Prove that the velocity of the liquid at P is

$$6fy\left\{\frac{a^5}{(a^4+f^2y^2)^{\frac{5}{2}}} - \frac{1}{(f^2+y^2)^{\frac{5}{2}}}\right\}.$$

18. Prove that

$$\phi = f(t)\left\{(r^2+a^2-2az)^{-\frac{1}{2}} + (r^2+a^2+2az)^{-\frac{1}{2}} - r^{-1}\right\} + \psi(t)$$

is the velocity potential of a liquid, and interpret it. Find the surfaces of equal pressure if gravity in the negative direction of the axis of z be the only force acting.

19. Liquid enters a right circular cylindrical vessel by a supply pipe at the centre O, and escapes by a pipe at a point A in the circumference; show that the velocity at any point P is proportional to $PB/PA \,.\, PO$, where B is the other end of the diameter AO. The vessel is supposed so shallow that the motion is in two dimensions.

CHAPTER II.

MOTION OF CYLINDERS AND SPHERES IN AN INFINITE LIQUID.

39. THE present chapter will be devoted to the consideration of certain problems of two-dimensional motion, and we shall also discuss the motion of a sphere in an infinite liquid.

If a right circular cylinder is moving in a liquid, the pressure of the liquid at any point of the cylinder passes through its axis, and therefore the resultant pressure of the liquid on the cylinder reduces to a single force, which can be calculated as soon as the pressure has been determined. Now the pressure at any point of the liquid is found by means of the equation

$$p/\rho + \phi + \tfrac{1}{2}q^2 = C,$$

and therefore p can be determined as soon as the velocity potential is known. Hence the first step towards the solution of problems of this character is to find the velocity potential.

If the cylinder is not circular, the resultant pressure of the liquid upon its surface will usually be reducible to a single force and a couple, and the problem becomes more complicated. The motion of cylinders, which are not circular, can be most conveniently treated by means of the dynamical methods explained in the next chapter. In the present chapter we shall show how to find the motion of an infinite liquid, in which cylinders of certain given forms are moving, and we shall also work out the solution of certain special problems relating to the motion of circular cylinders and spheres.

40. If the liquid be at rest, and a cylinder of any given form be set in motion in any manner, the subsequent motion of the liquid will be irrotational and acyclic, and is therefore completely determined by means of a velocity potential. It is however more convenient to employ Earnshaw's current function ψ. This function, when the motion is irrotational, satisfies the equation

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0 \quad\text{.......................(1)}$$

at all points of the liquid.

The integral[1] of this equation is

$$\psi = f(x + \iota y) + F(x - \iota y) \quad\text{.................(2),}$$

also
$$u = -\frac{d\psi}{dy}, \qquad v = +\frac{d\psi}{dx} \quad\text{.......................(3).}$$

We must now consider the boundary conditions to be satisfied by ψ.

If the liquid is at rest at infinity (which will usually be the case), $d\psi/dx$ and $d\psi/dy$ must vanish at infinity. If any portions of the boundary consist of fixed surfaces, the normal component of the velocity must vanish at such fixed boundaries, and therefore the fixed boundaries must coincide with a stream line. This requires that $\psi =$ const. at all points of fixed boundaries.

When the cylindrical boundary is in motion, the component velocity of the liquid along the normal, must be equal to the component velocity of the cylinder in the same direction.

(i) Let the cylinder be moving with velocity U parallel to the axis of x, and let θ be the angle which the normal to the cylinder makes with this axis; then at the surface

$$u \cos\theta + v \sin\theta = U \cos\theta.$$

Now $\cos\theta = dy/ds$; $\sin\theta = -dx/ds$; therefore by (3)

$$\frac{d\psi}{ds} = U\frac{dy}{ds}.$$

Integrating along the boundary, we obtain

$$\psi = Uy + A \quad\text{.........................(4),}$$

where A is a constant.

[1] The easiest way of showing that (2) is a solution of (1), is to differentiate the right-hand side of (2) twice with respect to x, and twice with respect to y and add. Since the result is zero, this shows that (2) *satisfies* (1); also since (2) contains *two arbitrary functions*, it is the most general solution that can be obtained.

(ii) If the cylinder be moving with velocity V parallel to the axis of y, it can be shown in the same manner that the surface condition is

$$\psi = -Vx + B \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

(iii) Let the cylinder be rotating with angular velocity ω; then at the surface

$$u \cos\theta + v \sin\theta = -\omega y \cos\theta + \omega x \sin\theta,$$

or

$$\frac{d\psi}{ds} = -\omega r \frac{dr}{ds}.$$

Therefore

$$\psi = -\tfrac{1}{2}\omega r^2 + C \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$.

When there are any number of moving cylinders in the liquid, conditions (4), (5) and (6) must be satisfied at the surfaces of each of the moving cylinders.

In addition to the surface conditions, ψ must satisfy the following conditions at every point of space occupied by the liquid; viz. ψ must be a function which is a solution of Laplace's equation (1), and which together with its first derivatives must be finite and continuous at every point of the liquid.

If we take any solution of (1), and substitute its value in (4), (5) or (6), we shall in many cases be able to determine the current function due to the motion of a cylinder, whose cross section is some curve, in one of the three prescribed manners.

In most of the applications which follow ψ will be of the form

$$\psi = f(x + \iota y) + f(x - \iota y) \quad\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

also by (3)

$$\frac{d\phi}{dx} = \frac{d\psi}{dy}, \quad \frac{d\phi}{dy} = -\frac{d\psi}{dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

From these equations we see that

$$\phi + \iota\psi = 2\iota f(x + \iota y) \quad\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

and therefore when ψ is known, ϕ can be found by equating the real and imaginary parts of (9).

Motion of a Circular Cylinder.

41. Let

$$\psi = -\tfrac{1}{2} Va^2 \left(\frac{1}{x + \iota y} + \frac{1}{x - \iota y}\right).$$

Transforming to polar coordinates, and using De Moivre's theorem, we obtain

$$\psi = - Va^2 x / r^2 \quad\text{......................(10)}.$$

When $r = a$, $\psi = - Vx$; equation (10) consequently determines the current function, when a circular cylinder of radius a is moving parallel to the axis of y, in an infinite liquid with velocity V.

By (9) the velocity potential is

$$\phi = - Va^2 y / r^2 \quad\text{......................(11)}.$$

42. Let us now suppose that the cylinder is of finite length unity, and that the liquid is bounded by two vertical parallel planes, which are perpendicular to the axis of the cylinder.

In order to find the motion when the cylinder is descending vertically under the action of gravity, let β be the distance of the axis of the cylinder at time t from some fixed point in its line of motion which we shall choose as the origin, and let (x, y) be the coordinates of any point of the liquid referred to the fixed origin, the axis of y being measured vertically downwards; also let (r, θ) be polar coordinates of the same point referred to the axis of the cylinder as origin. By (11)

$$\phi = -\frac{Va^2}{r} \sin\theta = -\frac{Va^2 (y - \beta)}{x^2 + (y - \beta)^2},$$

and therefore since $\qquad d\beta/dt = V,$

$$\dot{\phi} = -\frac{a^2 \dot{V}}{r} \sin\theta + \frac{a^2 V^2}{r^2} - \frac{2a^2 V^2}{r^2} \sin^2\theta,$$

and therefore at the surface, where $r = a$,

$$\dot{\phi} = - a\dot{V} \sin\theta + V^2 \cos 2\theta.$$

Also $\qquad q^2 = \left(\frac{d\phi}{dr}\right)^2 + \left(\frac{1}{r}\frac{d\phi}{d\theta}\right)^2;$

therefore when $\qquad r = a,$

$$q^2 = V^2.$$

Whence

$$p/\rho = a\dot{V}\sin\theta - V^2\cos 2\theta - \tfrac{1}{2}V^2 + g(\beta + a\sin\theta) + C \ldots (12).$$

The horizontal component of the pressure is evidently zero; the vertical component is

$$Y = -a\int_0^{2\pi} p\sin\theta d\theta.$$

Substituting the value of p from (12) and integrating, we obtain

$$Y = -\pi\rho a^2(\dot{V} + g).$$

Hence if σ be the density of the cylinder, the equation of motion is

$$\pi\sigma a^2\dot{V} = Y + \pi\sigma g a^2,$$

or

$$(\sigma + \rho)\dot{V} = (\sigma - \rho)g \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Integrating this equation, we obtain

$$V = v + \frac{(\sigma - \rho)gt}{(\sigma + \rho)},$$

where v is the initial velocity measured vertically downwards.

We therefore see that the cylinder will move in a vertical straight line, with a constant acceleration which is equal to

$$g(\sigma - \rho)/(\sigma + \rho).$$

In order to pass to the case in which there are no forces in action, we must put $g = 0$, in which case V remains constant and equal to its initial value. It thus appears that the only effect of the liquid is to produce an apparent increase in the inertia of the cylinder, which is equal to the mass of the liquid displaced.

By combining these two results, we see that if the cylinder be projected in any manner under the action of gravity, it will describe a parabola with vertical acceleration $g(\sigma - \rho)/(\sigma + \rho)$.

It is well known that if a solid body be projected in a liquid of unlimited extent, and no impressed forces are in action, it will not continue to move with constant velocity, but will gradually come to rest. One reason of this discrepancy is, that we have proceeded upon the supposition that the liquid is frictionless, whereas all liquids with which we are acquainted are more or less viscous, and the viscosity produces a gradual conversion of kinetic energy into heat. We shall consider this question more fully when discussing the motion of a sphere.

43. The motion of a cylinder in a liquid, which is bounded by a fixed external cylinder, is a problem of considerable difficulty. If however the cylinders are initially concentric, the initial motion can easily be found; and this problem will afford an example of the use of the velocity potential.

The velocity potential, as we know, satisfies Laplace's equation, which when transformed[1] into polar coordinates becomes

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2} = 0 \text{..................(14)}.$$

Let us endeavour to satisfy this equation by assuming $\phi = F(r)\,\epsilon^{\iota n\theta}$; this will be possible, provided

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{n^2F}{r^2} = 0.$$

Assuming $F = r^m$, the equation reduces to $m^2 - n^2 = 0$, whence $m = \pm n$, and therefore the required solution is

$$\phi = (Ar^n + Br^{-n})\,\epsilon^{\iota n\theta} \text{....................(15)}.$$

In this solution n may have any value whatever, and the real and imaginary parts of the above expression will be independent solutions of (14).

Let us now suppose that the radius of the outer cylinder, which is supposed to be fixed, is c; and let the inner one be started with velocity U.

Since the velocity of the liquid at the surface of the outer cylinder must be wholly tangential, the boundary condition is

$$\frac{d\phi}{dr} = 0, \text{ when } r = c \text{ ....(16)}.$$

At the surface of the inner cylinder, which is moving with velocity U, the component velocities of the cylinder and liquid along the radius must be equal; whence the boundary condition at the inner cylinder is

$$\frac{d\phi}{dr} = U\cos\theta, \text{ when } r = a \text{ ....(17)},$$

θ being measured from the direction of U.

If in (15) we put $n = 1$, the function

$$\phi = (Ar + B/r)\cos\theta \text{....................(18)}$$

[1] This transformation can be most easily effected, by forming the equation of continuity in polar coordinates.

is a solution of Laplace's equation; if therefore we can determine A and B so as to satisfy (16) and (17), the problem will be solved.

Substituting from (18) in (16) we obtain

$$Ac^2 - B = 0.$$

Substituting in (17) we obtain

$$Aa^2 - B = Ua^2.$$

Solving and substituting in (18) we obtain

$$\phi = -\frac{Ua^2}{c^2 - a^2}\left(r + \frac{c^2}{r}\right)\cos\theta.$$

If we put $c = \infty$, we fall back on our previous result of a cylinder moving in an infinite liquid.

We can now determine the impulsive force which must be applied to the inner cylinder, in order to start it with velocity U.

By § 25, equation (39), it follows that if a liquid which is at rest be set in motion by means of an impulse, and ϕ be the velocity potential of the initial motion, the impulsive pressure at any point of the liquid is equal to $-\rho\phi$.

Hence if M be the mass of the cylinder, F the impulse, the equation of motion is

$$\begin{aligned} MU &= F - \int_0^{2\pi} p\cos\theta\, d\theta \\ &= F + \rho a\int_0^{2\pi}\phi\cos\theta\, d\theta \\ &= F - \frac{U\pi\rho a^2(c^2 + a^2)}{c^2 - a^2}, \end{aligned}$$

whence since $M = \pi\sigma a^2$

$$F = \left\{1 + \frac{\rho(c^2 + a^2)}{\sigma(c^2 - a^2)}\right\} MU.$$

The Lemniscate of Bernoulli.

44. The lemniscate of Bernoulli is a bicircular quartic curve whose equation in Cartesian coordinates is $(x^2 + y^2)^2 = 2c^2(x^2 - y^2)$, or in polar coordinates $r^2 = 2c^2\cos 2\theta$. In order to find the current function when a cylinder, whose cross section is this curve, is moving parallel to x in an infinite liquid, let us put $u = x + \iota y$, $v = x - \iota y$, and assume

$$\psi_x = \tfrac{1}{2}Uc\iota\left\{\frac{u}{(u^2 - c^2)^{\frac{1}{2}}} - \frac{v}{(v^2 - c^2)^{\frac{1}{2}}}\right\} \quad \text{............(19).}$$

Now $$u^2 - c^2 = r^2(\cos 2\theta + \iota \sin 2\theta) - c^2;$$
whence at the surface where $r^2 = 2c^2 \cos 2\theta$, the right-hand side becomes
$$2c^2 \cos^2 2\theta + \iota c^2 \sin 4\theta - c^2 = c^2(\cos 2\theta + \iota \sin 2\theta)^2;$$
whence $$\frac{u}{(u^2 - c^2)^{\frac{1}{2}}} = \frac{r(\cos\theta + \iota \sin\theta)}{c(\cos 2\theta + \iota \sin 2\theta)} = r(\cos\theta - \iota\sin\theta)/c.$$

Therefore at the surface
$$\psi_x = Ur\sin\theta = Uy.$$

The value of ψ_x given by (19), is therefore the current function due to the motion parallel to x with velocity U, of a cylinder whose cross section is a lemniscate of Bernoulli.

If we put $$\psi_y = -\tfrac{1}{2}Vc\left\{\frac{u}{(u^2 - c^2)^{\frac{1}{2}}} + \frac{v}{(v^2 - c^2)^{\frac{1}{2}}}\right\},$$
$$\psi_3 = -\tfrac{1}{2}\omega c^3\left\{\frac{1}{(u^2 - c^2)^{\frac{1}{2}}} + \frac{1}{(v^2 - c^2)^{\frac{1}{2}}}\right\},$$
it can be shown in a similar manner, that ψ_y is the current function, when a cylinder of this form is moving parallel to y with velocity V; and that ψ_3 is the current function, when the cylinder is rotating with angular velocity ω about its axis.

If the cross section be the cardioid $r = 2c(1 + \cos\theta)$, the values of ψ_x and ψ_y can be obtained by writing $(u^{\frac{1}{2}} - c^{\frac{1}{2}})^2$, $(v^{\frac{1}{2}} - c^{\frac{1}{2}})^2$ for $(u^2 - c^2)^{\frac{1}{2}}$, $(v^2 - c^2)^{\frac{1}{2}}$ in the preceding formulae; but the value of ψ_3 can not be so simply obtained. See *Quart. Jour.* vol. XX. p. 246.

An Equilateral Triangle.

45. The preceding methods may also be employed, to find the motion of a liquid contained within certain cylindrical cavities, which are rotating about an axis.

Let $$\psi = \tfrac{1}{2}A\{(x + \iota y)^3 + (x - \iota y)^3\}$$
$$= A(x^3 - 3xy^2) = Ar^3\cos 3\theta.$$

Substituting in (6), the boundary condition becomes
$$A(x^3 - 3xy^2) + \tfrac{1}{2}\omega(x^2 + y^2) = C \qquad \text{.............(20)}.$$

If we choose the constants so that the straight line $x = a$, may form part of the boundary, we find
$$A = \frac{\omega}{6a}; \quad C = \frac{2\omega a^2}{3}.$$

Hence (20) splits up into the factors

$$(x-a);\ x+y\sqrt{3}+2a;\ x-y\sqrt{3}+2a.$$

The boundary therefore consists of three straight lines forming an equilateral triangle, whose centre of inertia is the origin.

Hence ψ is the current function due to liquid contained in an equilateral prism, which is rotating with angular velocity ω about an axis through the centre of inertia of its cross section. The values of ψ and ϕ, when cleared of imaginaries, are

$$\psi=\frac{\omega}{6a}r^3\cos 3\theta,\quad \phi=\frac{\omega}{6a}r^3\sin 3\theta.$$

An Elliptic Cylindrical Cavity.

46. Let $$\psi=\tfrac{1}{2}A\{(x+\iota y)^2+(x-\iota y)^2\}$$
$$=A(x^2-y^2).$$

Substituting in (6) we find

$$A(x^2-y^2)+\tfrac{1}{2}\omega(x^2+y^2)=C.$$

Putting $$\frac{\omega+2A}{2C}=\frac{1}{a^2};\quad \frac{\omega-2A}{2C}=\frac{1}{b^2},$$

the equation of the boundary becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and $$\psi=-\frac{\omega(a^2-b^2)}{2(a^2+b^2)}(x^2-y^2),$$

ψ is therefore the current function due to the motion of liquid contained in an elliptic cylinder, which is rotating about its axis.

Elliptic Cylinder.

47. The problem of finding the motion of an elliptic cylinder in an infinite liquid cannot be solved by such simple methods as the foregoing; in order to effect the solution we require to employ the method of Conjugate Functions.

Def. *If ξ and η are functions of x and y, such that*

$$\xi+\iota\eta=f(x+\iota y)\ \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

then ξ and η are called conjugate functions of x and y.

If we differentiate (21) first with respect to x, and afterwards

with respect to y, eliminate the arbitrary function, and then equate the real and imaginary parts, we shall obtain the equations

$$\frac{d\xi}{dx}=\frac{d\eta}{dy},\quad \frac{d\xi}{dy}=-\frac{d\eta}{dx} \qquad (22).$$

Comparing these equations with (8), we see that ϕ and ψ are conjugate functions of x and y.

From equations (22) we also see that

$$\frac{d\xi}{dx}\frac{d\eta}{dx}+\frac{d\xi}{dy}\frac{d\eta}{dy}=0 \qquad (23),$$

$$\nabla^2\xi=0,\quad \nabla^2\eta=0 \qquad (24).$$

Equation (23) shows that the curves $\xi=$ const., $\eta=$ const., form an orthogonal system; and equations (24) show that ξ and η each satisfy Laplace's equation.

If ϕ and ψ are conjugate functions of x and y, and ξ and η are also conjugate functions of x and y, then ϕ and ψ are conjugate functions of ξ and η.

For
$$\phi+\iota\psi=F(x+\iota y),$$
and
$$\xi+\iota\eta=f(x+\iota y),$$
whence eliminating $x+\iota y$, we have
$$\phi+\iota\psi=\chi(\xi+\iota\eta).$$

From this proposition combined with (24), it follows that if the equation $\nabla^2\psi=0$ be transformed by taking ξ and η as independent variables

$$\frac{d^2\psi}{d\xi^2}+\frac{d^2\psi}{d\eta^2}=0 \qquad (25).$$

48. We can now find the current function due to the motion of an elliptic cylinder.

Let
$$\begin{aligned} x+\iota y &= c\cos(\xi-\iota\eta) \\ &= c\cos\xi\cosh\eta+\iota c\sin\xi\sinh\eta, \end{aligned}$$
then
$$\left.\begin{aligned} x&=c\cos\xi\cosh\eta \\ y&=c\sin\xi\sinh\eta \end{aligned}\right\} \qquad (26),$$
whence the curves $\eta=$ const., $\xi=$ const., represent a family of confocal ellipses and hyperbolas, the distance between the foci being $2c$.

If a and b be the semi-axes of the cross section of the elliptic cylinder $\eta=\beta$, then,

$$a=c\cosh\beta,\quad b=c\sinh\beta.$$

If β is exceedingly large, $\sinh\beta$ and $\cosh\beta$ both approximate to the value $\frac{1}{2}c\epsilon^{\beta}$; and therefore as the ellipse increases in size, it approximates to a circle whose radius is $\frac{1}{2}c\epsilon^{\beta}$.

It can be verified by trial, that (25) can be satisfied by a series of terms of the form $\epsilon^{-n\eta}(A_n \cos n\xi + B_n \sin n\xi)$; and if n be a positive quantity not less than unity, this is the proper form of ψ outside an elliptic cylinder, since it continually diminishes as η increases.

When the cylinder is moving parallel to its major axis with velocity U, let us assume

$$\psi_x = A\epsilon^{-\eta}\sin\xi.$$

Substituting in (4) we obtain

$$A\epsilon^{-\beta}\sin\xi = Uc\sinh\beta\sin\xi + C,$$

where $\eta=\beta$ is the equation of the cross section of the cylinder.

Since this equation is to be satisfied at every point of the boundary, we must have $C=0$, $A=Uc\epsilon^{\beta}\sinh\beta$; whence

$$\psi_x = Uc\epsilon^{-\eta+\beta}\sinh\beta\sin\xi \quad\text{..............} (27).$$

When the cylinder is moving parallel to its minor axis with velocity V, it may be shown in the same manner that

$$\psi_y = -Vc\epsilon^{-\eta+\beta}\cosh\beta\cos\xi \quad\text{..............} (28).$$

Lastly let us suppose that the cylinder is rotating with angular velocity ω about its axis. Then

$$x^2+y^2 = c^2(\cos^2\xi\cosh^2\eta + \sin^2\xi\sinh^2\eta)$$
$$= \tfrac{1}{2}c^2(\cosh 2\eta + \cos 2\xi).$$

Let us therefore assume

$$\psi_3 = B\epsilon^{-2\eta}\cos 2\xi.$$

Substituting in (6) we obtain

$$B\epsilon^{-2\beta}\cos 2\xi + \tfrac{1}{4}\omega c^2(\cosh 2\beta + \cos 2\xi) = C,$$

whence $$B = -\tfrac{1}{4}\omega c^2\epsilon^{2\beta},\quad C = \tfrac{1}{4}\omega c^2\cosh 2\beta,$$

and therefore $$\psi_3 = -\tfrac{1}{4}\omega c^2\epsilon^{-2(\eta-\beta)}\cos 2\xi \quad\text{..............} (29).$$

49. If we suppose that $\beta=0$, the ellipse degenerates into a straight line joining the foci, and (28) becomes

$$\psi_y = -Vc\epsilon^{-\eta}\cos\xi \quad\text{....................} (30).$$

It might therefore be supposed that (30) gives the value of the current function, due to a lamina of breadth $2c$, which moves with

velocity V, perpendicularly to itself. This however is not the case, inasmuch as the velocity at the edges of the lamina becomes infinite, and therefore the solution fails. To prove this, we have

$$\frac{d\psi}{d\eta}=\frac{d\psi}{dx}\frac{dx}{d\eta}+\frac{d\psi}{dy}\frac{dy}{d\eta}$$

$$=c\sinh\eta\cos\xi\frac{d\psi}{dx}+c\cosh\eta\sin\xi\frac{d\psi}{dy};$$

and
$$\frac{d\psi}{d\xi}=-c\cosh\eta\sin\xi\frac{d\psi}{dx}+c\sinh\eta\cos\xi\frac{d\psi}{dy},$$

whence squaring and adding, we obtain

$$c^2(\sinh^2\eta\cos^2\xi+\cosh^2\eta\sin^2\xi)\,q^2=\left(\frac{d\psi}{d\eta}\right)^2+\left(\frac{d\psi}{d\xi}\right)^2=V^2c^2\epsilon^{-2\eta}\ldots(31).$$

The coordinates of an edge are $x=\pm c$, $y=0$; and therefore in the neighbourhood of an edge η and ξ are very small quantities; and therefore by (31) the velocity in the neighbourhood of an edge is

$$q=\frac{V}{(\eta^2+\xi^2)^{\frac{1}{2}}},$$

which becomes infinite at the edge itself, where η and ξ are zero. It therefore follows that the pressure in the neighbourhood of an edge is negative, which is physically impossible.

Since the pressure is positive at a sufficient distance from the edge, there will be a surface of zero pressure dividing the regions of positive and negative pressures; and it might be thought that the interpretation of the formulae would be, that a hollow space exists in the liquid surrounding the edges, which is bounded by a surface of zero pressure. But the condition that a free surface should be a surface of zero (or constant) pressure, although a necessary one, is not sufficient; it is further necessary, that such a surface should be a surface of no flux, which satisfies the kinematical condition of a bounding surface § 12, equation (17); and it will be found on investigating the question, that no surface exists, which is a surface of zero (or constant) pressure, and at the same time satisfies the conditions of a bounding surface. The solution altogether fails in the case of a lamina.

When the velocity of the solid is constant and equal to V, the easiest way of dealing with a problem of this character is to reverse the motion by supposing the solid to be at rest, and that the liquid flows past it, the velocity at infinity being equal to $-V$.

The correct solution in the case of a lamina has been given by Kirchhoff[1], and he has shown that behind the lamina there is a region of *dead water*, i.e. water at rest, which is separated from the remainder of the liquid by two surfaces of discontinuity, which commence at the two edges of the lamina, and proceed to infinity in the direction in which the stream is flowing. Since the liquid on one side of this surface of discontinuity is at rest, its pressure is constant; and therefore since the motion is steady, the pressure, and therefore the velocity of the moving liquid, must be constant at every point of the surface of discontinuity. It may be added that a surface of discontinuity is an imaginary surface described in the liquid, such that the tangential component of the velocity suddenly changes as we pass from one side of the surface to the other.

Motion of a Sphere.

50. The determination of the velocity potential, when a solid body of any given shape is moving in an infinite liquid, is one of great difficulty, and the only problem of the kind which has been completely worked out is that of an ellipsoid, which of course includes a sphere as a particular case.

We shall however find it simpler in the case of a sphere to solve the problem directly, which we shall proceed to do.

Let the sphere be moving along a straight line with velocity V, and let (r, θ, ω) be polar coordinates referred to the centre of the sphere as origin, and to the direction of motion as initial line. The conditions of symmetry show that ϕ must be a function of (r, θ) and not of ω, hence by § 7, equation (12), the equation of continuity is

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2} + \frac{1}{r^2}\cot\theta\frac{d\phi}{d\theta} = 0 \quad\text{.........}\quad (32).$$

The boundary condition, which expresses the fact that the normal component of the velocity of the liquid at the surface of the sphere is equal to the normal component of the velocity of the sphere itself, is

$$\frac{d\phi}{dr} = V\cos\theta \quad\text{.......................}\quad (33).$$

Equation (33) suggests that ϕ must be of the form $F(r)\cos\theta$; we shall therefore try whether we can determine F so as to satisfy

[1] See also, Michell, *Phil. Trans.* 1890, p. 389; Love, *Proc. Camb. Phil. Soc.* vol. VII. p. 175.

(32). Substituting this value of ϕ, we find that (32) will be satisfied, provided

$$\frac{d^2F}{dr^2}+\frac{2}{r}\frac{dF}{dr}-\frac{2F}{r^2}=0 \quad\text{.................}\quad (34).$$

To solve (34), assume $F=r^m$; whence on substitution we obtain

$$(m-1)(m+2)=0;$$

which requires that $m=1$ or -2.

A particular solution of (32) is therefore

$$\phi=(Ar+Br^{-2})\cos\theta.$$

Since the liquid is supposed to be at rest at infinity, $d\phi/dr=0$ when $r=\infty$, and therefore $A=0$. To find B, substitute in (33) and put $r=a$, and we find

$$A=-\tfrac{1}{2}Va^3,$$

whence
$$\phi=-\frac{Va^3\cos\theta}{2r^2} \quad\text{....................}\quad (35).$$

This is the expression for the velocity potential due to the motion of a sphere in an infinite liquid.

In order to determine the motion, when a sphere is descending vertically under the action of gravity, let γ be the distance of its centre at time t from some fixed point in its line of motion, which we shall choose as the origin; let the axis of z be measured vertically downwards and let x, y, z be the coordinates of any point of the liquid referred to the *fixed* origin.

By (35)
$$\phi=-\frac{Va^3(z-\gamma)}{2\{x^2+y^2+(z-\gamma)^2\}^{\frac{3}{2}}},$$

and therefore since $\dot{\gamma}=V$,

$$\frac{d\phi}{dt}=-\frac{\dot{V}a^3\cos\theta}{2r^2}+\frac{V^2a^3}{2r^3}-\frac{3V^2a^3\cos^2\theta}{2r^3};$$

and therefore at the surface where $r=a$,

$$\frac{d\phi}{dt}=-\tfrac{1}{2}\dot{V}a\cos\theta-\tfrac{1}{2}V^2(3\cos^2\theta-1),$$

also
$$q^2=\left(\frac{d\phi}{dr}\right)^2+\left(\frac{1}{r}\frac{d\phi}{d\theta}\right)^2$$
$$=V^2(\cos^2\theta+\tfrac{1}{4}\sin^2\theta),$$

and therefore

$$p/\rho=C+g(\gamma+a\cos\theta)+\tfrac{1}{2}\dot{V}a\cos\theta+\tfrac{1}{8}V^2(9\cos^2\theta-5)\text{...}(36).$$

If Z be the force due to the pressure of the liquid, which opposes the motion,

$$\begin{aligned} Z &= \iint p \cos\theta \, dS \\ &= 2\pi a^2 \int_0^\pi p \cos\theta \sin\theta \, d\theta \\ &= \tfrac{4}{3}\pi\rho a^3 \left(\tfrac{1}{2}\dot{V} + g\right) \quad \text{..........................(37)} \end{aligned}$$

by (36). If therefore σ be the density of the sphere, the equation of motion is

$$\tfrac{4}{3}\pi\sigma a^3 \dot{V} = -\tfrac{4}{3}\pi\rho a^3 \left(\tfrac{1}{2}\dot{V} + g\right) + \tfrac{4}{3}\pi a^3 \sigma g$$

or

$$(\sigma + \tfrac{1}{2}\rho)\dot{V} = (\sigma - \rho) g \text{.....................(38).}$$

Hence the sphere descends with vertical acceleration

$$g(\sigma - \rho)/(\sigma + \tfrac{1}{2}\rho).$$

In order to pass to the case in which the sphere is projected with a given velocity and no forces are in action, we must put $g = 0$, and we see that $V = \text{const.} = \text{its initial value}$; hence the sphere continues to move with its velocity of projection, and the effect of the liquid is to produce an apparent increase in the inertia of the sphere, which is equal to half the mass of the liquid displaced.

It also follows that if the sphere be projected in any manner under the action of gravity, it will describe a parabola with vertical acceleration $g(\sigma - \rho)/(\sigma + \tfrac{1}{2}\rho)$.

51. Let us now suppose that the sphere is moving with constant velocity V under the action of no forces. The equation determining the pressure is

$$\frac{p}{\rho} = C + V\frac{d\phi}{dz} - \tfrac{1}{2}q^2.$$

Since $d\phi/dz$ and q vanish at infinity, it follows that $C = \Pi/\rho$, where Π is the pressure at infinity, whence

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + V\frac{d\phi}{dz} - \tfrac{1}{2}q^2;$$

and therefore at the surface,

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \tfrac{1}{8}V^2(9\cos^2\theta - 5).$$

The right-hand side of this equation will be a minimum when $\theta = \tfrac{1}{2}\pi$, in which case it becomes $\Pi/\rho - \tfrac{5}{8}V^2$. Hence if

$$\Pi < \tfrac{5}{8}V^2\rho$$

the pressure will become negative within a certain region in the neighbourhood of the equator, and the solution fails. When V exceeds the critical value $(8\Pi/5\rho)^{\frac{1}{2}}$ it is probable that a region of dead water exists behind the sphere, which is separated from the rest of the liquid by a vortex sheet.

52. In discussing the motion of a cylinder, we found that if the solid were projected in a liquid and no forces were in action, the solid would continue to move in a straight line with its original velocity of projection; and we called attention to the fact that this result was contrary to experience; and that one reason of this discrepancy between theory and observation arose from the fact that all liquids are more or less viscous, the result of which is that kinetic energy is gradually converted into heat. The motion of viscous fluids is beyond the scope of an elementary work such as the present, but a few remarks on the subject will not be out of place.

Let us suppose that fluid is moving in strata parallel to the plane xy, with a variable velocity U, which is parallel to the axis of x. Let U be the velocity of the stratum AB, $U+\delta U$ of the stratum CD, and let δz be the distance between AB and CD.

If the fluid were frictionless, the action between the fluid on either side of the plane AB would be a hydrostatic pressure p, whose direction is perpendicular to this plane, and consequently no tangential action or shearing stress could exist; if however the fluid is viscous, the action between the fluid on either side of the plane AB, usually consists of an oblique pressure (or tension), and may therefore be resolved into a normal component *perpendicular* to the plane, and a tangential component *in* the plane.

The usual theory of viscosity supposes, that if F be the tangential stress on AB per unit of area, $F+\delta F$ the corresponding stress on CD, then the latter stress is proportional to the relative velocity of the two strata divided by the distance between the strata, so that

$$F+dF \propto \frac{U+\delta U-U}{\delta z},$$

whence proceeding to the limit

$$F \propto \frac{dU}{dz}.$$

We may therefore put $F = \mu \dfrac{dU}{dz}$ ....(39),

where μ is a constant. The constant μ is called the *viscosity*; it is a numerical quantity whose value is different for different fluids, and also depends upon the temperature.

The viscosity is a quantity which corresponds to the *rigidity* in the Theory of Elasticity. If a shearing stress F be applied parallel to the axis of x, and in a plane parallel to the plane xy, to an elastic solid, it is known from the Theory of Elasticity, that

$$F = n d\alpha/dz,$$

where α is the displacement parallel to x. Whence the ratio of the shearing stress F, to the shearing strain $d\alpha/dz$ produced by it, is equal to a constant n, which is called the *rigidity*. Now in the hydrodynamical theory of viscous fluids, dU/dz is equal to the rate at which shearing strain is produced by the shearing stress F; hence (39) asserts that the ratio of the shearing stress to the rate at which shearing strain is produced, is equal to a constant μ, which is called the viscosity.

If the shearing stress F is applied in the plane $z = c$, and if $U = uz/c$, (39) becomes

$$F = \mu u/c \text{..........................(40)},$$

where u is the velocity of the fluid in the plane $z = c$. Hence if $u = 1$, and $c = 1$, then $F = \mu$. We may therefore define the viscosity as follows[1].

The viscosity is equal to the tangential force per unit of area, on either of two parallel planes at the unit of distance apart, one of which is fixed, whilst the other moves with the unit of velocity, the space between being filled with the viscous fluid.

Equation (40) shows that the dimensions of μ are $[ML^{-1}T^{-1}]$.

If we put $\nu = \mu/\rho$, where ρ is the density, the quantity ν is called the *kinematic coefficient of viscosity*. The dimensions of ν are $[L^2T^{-1}]$.

The equations of motion of a viscous fluid are known, and the motion of a sphere which is descending under the action of gravity in a slightly viscous *liquid*, such as water, has been worked out by myself; and I have shown that if the sphere be initially projected downwards with velocity V, its velocity at any subsequent time will

[1] Maxwell's *Heat*, p. 298, fourth edition.

be approximately given by the equation

$$v = f(1-\epsilon^{-\lambda t})/\lambda + V\epsilon^{-\lambda t}$$

where $$f = \frac{(\sigma-\rho)g}{\sigma+\frac{1}{2}\rho}, \quad \lambda = \frac{9\mu}{a^2(2\sigma+\rho)}.$$

If no forces are in action, the velocity at time t is

$$v = V\epsilon^{-\lambda t},$$

which shows that the velocity diminishes with the time.

If the liquid were frictionless, μ would be zero, and we should fall back on our previous results.

53. The resistance, which a ship experiences in moving through water, is principally due to the following three causes, (i) viscosity, (ii) skin friction, (iii) wave resistance.

The first has been already considered, and since the viscosity of water is a small quantity, viz. ·014 dynes per square centimetre in C.G.S. units, the temperature being 24·5° C.; it appears that the effect of viscosity is small.

The second cause is due to the friction between the sides of the vessel and the water in contact with it, and in the opinion of Mr Froude[1] is the principal cause of the resistance experienced by ships.

The third cause is due to the fact that, when a ship is in motion on a river or in the open sea, waves are continually being generated, which require an expenditure of energy for their production, and this is necessarily supplied by the mechanical power which is employed to propel the ship. If therefore the ship were set in motion in a frictionless liquid and left to itself, the initial energy would gradually be dissipated in forming waves, and would be carried away by them, so that this cause alone would ultimately bring the ship to rest.

54. Having made this digression upon viscosity and resistance, we must return to the subject of this chapter.

Let us suppose that a spherical pendulum, which is surrounded with liquid, is performing small oscillations in a vertical plane.

Let $l-a$ be the length of the pendulum rod, whose mass we shall suppose small enough to be neglected, and let M' be the mass of the liquid displaced.

[1] On stream lines in relation to the resistance of ships. *Nature*, Vol. III.

If V be the horizontal velocity of the pendulum, it follows from (37), that the horizontal and vertical forces due to the pressure of the liquid are

$$X = \tfrac{1}{2} M' \dot{V}, \quad Y = M'g.$$

Now $V = a\dot{\theta}$, whence taking moments about the point of suspension, we obtain

$$M(\tfrac{2}{5}a^2 + l^2)\ddot{\theta} = -Mgl\theta - Xl + Yl\theta,$$

or
$$\{M(\tfrac{2}{5}a^2 + l^2) + \tfrac{1}{2}M'l^2\}\ddot{\theta} + (M - M')gl\theta = 0.$$

Whence if T be the time of a small oscillation, we have

$$T = 2\pi \sqrt{\frac{\tfrac{2}{5}Ma^2 + (M + \tfrac{1}{2}M')l^2}{(M - M')gl}} \quad \text{.................} (41).$$

If in (41) we put $\rho = 0$, or $M' = 0$, we shall obtain the period of a pendulum which is vibrating in vacuo, and (41) shows that the effect of the liquid is to increase the period.

This result is in accordance with a general dynamical principle (to which however there are certain exceptions), *that when a dynamical system is subject to a constraint which is equivalent to an increase in the inertia of the system, the periods of oscillation are usually greater than when the system is free.*

55. In the last example we have supposed that the liquid extends to infinity in all directions. In practice this is impossible, and it is therefore desirable to ascertain what effect the vessel which contains the liquid, produces on the period.

Let us therefore suppose that the sphere and liquid are contained within a rigid fixed spherical envelop of radius c, whose centre coincides with the equilibrium position of the sphere. The vertical pressure of the liquid upon the sphere is evidently equal to $M'g$; and the horizontal pressure will be of the form

$$\dot{V}F(x) + V^2 f(x),$$

where x is the distance of the centre of the sphere at time t from its mean position, and F and f are unknown functions. The moment of the latter about the point of suspension is

$$l\{\dot{V}F(x) + V^2 f(x)\};$$

and since in all problems relating to small oscillations, the squares and products of small quantities are neglected, it follows by Maclaurin's theorem that the moment is $l\dot{V}F(0)$, and therefore may be calculated on the supposition that the spheres are concentric. We must therefore first obtain the velocity potential

when the two spheres are concentric, and the inner sphere is set in motion with velocity V.

We have already shown that a solution of Laplace's equation is

$$\phi = (Ar + Br^{-2}) \cos \theta.$$

The boundary condition at the moving sphere is

$$\frac{d\phi}{dr} = V \cos \theta \quad \text{.......................(42)},$$

when $r = a$.

The boundary condition at the fixed envelop is

$$\frac{d\phi}{dr} = 0 \quad \text{..............................(43)},$$

when $r = c$. Substituting the above value of ϕ in (42) and (43), we obtain

$$A - 2Ba^{-3} = V, \quad A - 2Bc^{-3} = 0;$$

whence
$$A = -\frac{Va^3}{c^3 - a^3}, \qquad B = -\frac{Va^3c^3}{2(c^3 - a^3)}$$

and
$$\phi = -\frac{Va^3}{c^3 - a^3}\left(r + \frac{c^3}{2r^2}\right) \cos \theta \quad \text{.................(44)}.$$

The pressure on the sphere may thus be obtained from our previous results by writing $\frac{1}{2} Va(c^3 + 2a^3)/(c^3 - a^3)$ for $\frac{1}{2} Va$; whence

$$X = \tfrac{1}{2} M' \dot{V} (c^3 + 2a^3)/(c^3 - a^3), \quad Y = M'g,$$

and the equation of motion is

$$\{M(\tfrac{2}{5}a^2 + l^2) + \tfrac{1}{2} M'l^2 (c^3 + 2a^3)/(c^3 - a^3)\} \ddot{\theta} + (M - M') gl\theta = 0.$$

This equation shows that the effect of the spherical envelop, which produces an additional constraint, is to increase the period; and that its action is equivalent to an increase in the inertia of the pendulum which is equal to

$$\tfrac{1}{2} M' (c^3 + 2a^3)/(c^3 - a^3).$$

EXAMPLES.

1. Prove that $\qquad \phi = \log \dfrac{(x+a)^2+y^2}{(x-a)^2+y^2}$

gives a possible motion in two dimensions. Find the form of the stream lines, and prove that the curves of equal velocity are lemniscates.

2. In the irrotational motion of a liquid, prove that the motion derived from it, by turning the direction of motion at each point in one direction through 90° without changing the velocity, will also be a possible irrotational motion, the conditions at the boundaries being altered so as to suit the new motion.

Discuss the motion obtained in this way from the preceding example.

3. Liquid is moving irrotationally in two dimensions, between the space bounded by the two lines $\theta = \pm \frac{1}{6}\pi$ and the curve $r^3 \cos 3\theta = a^3$. The bounding curves being at rest, prove that the velocity potential is of the form

$$\phi = r^3 \sin 3\theta.$$

4. The space between the elliptic cylinder $(x/a)^2 + (y/b)^2 = 1$, and a similarly situated and coaxial cylinder bounded by planes perpendicular to the axis, is filled with liquid, and made to rotate with angular velocity ω about ~~a fixed~~ axis. Prove that the velocity potential with reference to the principal axes of the cylinder is $\omega xy(a^2 - b^2)/(a^2 + b^2)$, and that the surfaces of equal pressure when the angular velocity is constant, are the hyperbolic cylinders

$$\frac{x^2}{3a^2+b^2} - \frac{y^2}{3b^2+a^2} = C.$$

5. If $\phi = f(x, y)$, $\psi = F(x, y)$ are the velocity potential and current function of a liquid, and if we write

$$x = f(\phi, \psi), \quad y = F(\phi, \psi)$$

and from these expressions find ϕ and ψ; prove that the new values of ϕ and ψ will be the velocity potential and current function of some other motion of a liquid.

Hence prove that if $\phi = x^2 - y^2$, $\psi = 2xy$, the transformation gives the motion of a liquid in the space bounded by two confocal and coaxial parabolic cylinders.

6. In example 4 prove that the paths of the particles relative to the cylinder are similar ellipses, and that the paths in space are similar to the pericycloid

$$x = (a+b)\cos\theta + (a-b)\cos\left(\frac{a+b}{a-b}\right)^2\theta,$$

$$y = (a+b)\sin\theta + (a-b)\sin\left(\frac{a+b}{a-b}\right)^2\theta.$$

7. Water is enclosed in a vessel bounded by the axis of y and the hyperbola $2(x^2 - 3y^2) + x + my = 0$, and the vessel is set rotating about the axis of z. Prove that

$$\phi = 2(3x^2y - y^3) + xy - \tfrac{1}{2}m(x^2 - y^2),$$
$$\psi = 2(x^3 - 3xy^2) + \tfrac{1}{2}(x^2 - y^2) + mxy.$$

8. The space between two confocal coaxial elliptic cylinders is filled with liquid which is at rest. Prove that if the outer cylinder be moved with velocity U parallel to the major axis, and the inner with relative velocity V in the same direction, the velocity potential of the initial motion will be

$$\phi = Uc\cosh\eta\cos\xi - Vc\frac{\cosh(\beta-\eta)}{\sinh(\beta-\alpha)}\sinh\alpha\cos\xi,$$

where $\eta = \beta$, $\eta = \alpha$ are the equations of the outer and inner cylinders respectively, and $2c$ the distance between their foci.

9. If in the last example the outer cylinder were to rotate with angular velocity Ω, and the inner with angular velocity ω, prove that initially

$$\phi = \tfrac{1}{4}\Omega c^2\frac{\cosh 2(\eta-\alpha)}{\sinh 2(\beta-\alpha)}\sin 2\xi - \tfrac{1}{4}\omega c^2\frac{\cosh 2(\beta-\eta)}{\sinh 2(\beta-\alpha)}\sin 2\xi.$$

10. The transverse section of a uniform prismatic vessel is of the form bounded by the two intersecting hyperbolas represented by the equations

$$\sqrt{2}(x^2-y^2) + x^2 + y^2 = a^2, \quad \sqrt{2}(y^2-x^2) + x^2 + y^2 = b^2.$$

If the vessel be filled with water and made to rotate with angular velocity ω about its axis, prove that the initial component velocities at any point (x, y) of the water will be

$$\frac{\omega}{a^2+b^2}\{2y^3 - 6x^2y + \sqrt{2}(a^2-b^2)y\}$$

$$-\frac{\omega}{a^2+b^2}\{2x^3 - 6xy^2 + \sqrt{2}(b^2-a^2)x\}$$

respectively.

11. In the midst of an infinite mass of liquid at rest is a sphere of radius a, which is suddenly strained into a spheroid of small ellipticity. Find the kinetic energy due to the motion of the liquid contained between the given surface, and an imaginary concentric spherical surface of radius c; and show that if this imaginary surface were a real bounding surface which could not be deformed, the kinetic energy in this case would be to that in the former case in the ratio

$$c^5(3a^5 + 2c^5) : 2(c^5 - a^5)^2.$$

12. The space between two coaxial cylinders is filled with liquid, and the outer is surrounded by liquid extending to infinity, the whole being bounded by planes perpendicular to the axis. If the inner cylinder be suddenly moved with given velocity, prove that the velocity of the outer cylinder to that of the inner, will be in the ratio

$$2b^2c^2\rho : \rho(a^2b^2 - a^2c^2 + b^4 + b^2c^2) + \sigma(a^2 - b^2)(b^2 - c^2),$$

where a and b are the external and internal radii of the outer cylinder, σ its density, c the radius of the inner cylinder and ρ the density of the liquid.

13. A solid cylinder of radius a immersed in an infinite liquid, is attached to an axis about which it can turn, whose distance from the axis of the cylinder is c, and oscillates under the action of gravity. Prove that the length of the simple equivalent pendulum is

$$\frac{\frac{1}{2}a^2 + c^2(1 + \rho/\sigma)}{c(1 - \rho/\sigma)},$$

σ and ρ being the densities of the cylinder and liquid.

14. Liquid of density ρ is contained between two confocal elliptic cylinders and two planes perpendicular to their axes. The lengths of the semi-axes of the inner and outer cylinders are $c \cosh \alpha$, $c \sinh \alpha$, $c \cosh \beta$, $c \sinh \beta$ respectively. Prove that if the outer cylinder be made to rotate about its axis with angular velocity Ω, the inner cylinder will begin to rotate with angular velocity

$$\frac{\Omega\rho \operatorname{cosech} 2(\beta - \alpha)}{\rho \coth 2(\beta - \alpha) + \frac{1}{2}\sigma \sinh 4\alpha},$$

where σ is the density of the cylinder.

15. A circular cylinder of mass M, whose centre of inertia is at a distance c from its axis, is projected in an infinite liquid under the action of gravity. Prove that the centre of inertia of the cylinder and the displaced liquid will describe a parabola, while the cylinder oscillates like a pendulum of length

$$\{(M+M')k^2 \text{[illegible]}\}/2M'c,$$

where M' is the mass of the liquid displaced, and k is the radius of gyration of the cylinder about its axis.

16. A cylinder of radius a is surrounded by a concentric cylinder of radius b, and the intervening space is filled with liquid. The inner cylinder is moved with velocity u, and the outer with velocity v along the same straight line; prove that the velocity potential is

$$\phi = \frac{b^2 v - a^2 u}{b^2 - a^2} r \cos\theta + \frac{(v-u)\,a^2 b^2 \cos\theta}{(b^2 - a^2)\,r}.$$

17. A long cylinder of given radius is immersed in a mass of liquid bounded by a very large cylindrical envelop. If the envelop be suddenly moved in a direction perpendicular to the cylinder with velocity V, the cylinder will begin to move with velocity $\frac{1}{2}V$, provided the density of the cylinder be three times that of the liquid.

CHAPTER III.

MOTION OF A SINGLE SOLID IN AN INFINITE LIQUID.

56. IN the previous chapter, we obtained expressions for the velocity potential and current function in several cases in which a solid was moving in an infinite liquid; and we also worked out several problems relating to the motion of a right circular cylinder and a sphere, by first calculating the pressure, and then by integration determining the resultant force exerted by the liquid upon the solid. This method, in the case of solids other than circular cylinders and spheres, is excessively laborious; and we shall devote the present chapter to developing a dynamical theory, which will enable us to dispense with this operation.

The solid and surrounding liquid constitute a single dynamical system, in which the pressure exerted by the latter upon the former is an unknown reaction arising from contiguous portions of the system. The momentum of the solid can be calculated by the ordinary methods of Rigid Dynamics; and it will be shown in the present chapter that the momentum of the liquid can be expressed in terms of the velocities and coordinates of the solid. Now the position and motion of the solid are completely determined by six coordinates; and since the Principle of Momentum furnishes six independent equations of motion, it follows that if this principle is applied to the system which is composed of the solid and the surrounding liquid, sufficient equations will be obtained for completely determining the motion.

The Principle of Momentum will not, however, enable us to determine the motion when the liquid is bounded by a fixed surface; for the pressure exerted by the boundary upon the liquid is one of the forces which must be included in the equations of

motion which are the analytical expressions for this principle. In cases of this kind, it is necessary to use some form of equations of motion which are deducible from the Principle of Energy. The most convenient course to pursue is to employ Lagrange's equations; for since the pressure exerted by a fixed boundary does no work upon the system, it cannot appear amongst the forces in Lagrange's equations. When there is more than one moving solid, Lagrange's equations are always employed.

The Principle of Momentum could also be employed to determine the motion of a single solid in a *viscous* liquid, provided the liquid extends to infinity in all directions. Lagrange's equations, on the other hand, could not be employed; since they are inapplicable to systems in which there is a conversion of mechanical energy into heat.

57. Before proceeding further, it will be desirable to call attention to two additional dynamical propositions which will be required hereafter; and we shall then prove an important analytical theorem due to Green. The propositions are:

I. *The work done by an impulse is equal to half the product of the impulse into the sum of the components in the direction of the impulse, of the initial and final velocities of the point at which it is applied.*

II. *If a dynamical system be set in motion by given impulses, the work done by the impulses is greater when the system is free than when it is subject to constraint.*

The second proposition, from the name of its discoverer, is known as Bertrand's Theorem; and the simplest way of proving it is by means of the mixed transformation of Lagrange's equations, the complete theory of which was first given by myself[1] in 1887.

Let the coordinates of a dynamical system be divided into two groups θ and χ; and let κ be the generalized momentum of type χ. Then it is known that the kinetic energy of the system can be expressed in the form

$$T = \mathfrak{T} + \mathfrak{K},$$

where $\mathfrak{T}$ is a homogeneous quadratic function of the velocities $\dot{\theta}$, and $\mathfrak{K}$ is a similar function of the momenta κ. Since impulsive

[1] *Proc. Camb. Phil. Soc.* Vol. VI. p. 121.

forces are equal to the momenta produced by them, it follows that if the free system is set in motion by means of impulsive forces of type χ, the kinetic energy will be given by the above equation; but if a constraint is introduced whereby all the coordinates θ are compelled to remain constant quantities, $\mathfrak{T}$ will be zero, and the kinetic energy will be equal to $\mathfrak{K}$: whence the kinetic energy produced by the impulsive forces is less when the system is subject to constraint than when it is free.

Green's Theorem.

58. *Let ϕ and ψ be any two functions, which throughout the interior of a closed surface S are single valued, and which together with their first and second derivatives, are finite and continuous at every point within S; then*

$$\iiint\left(\frac{d\phi}{dx}\frac{d\psi}{dx}+\frac{d\phi}{dy}\frac{d\psi}{dy}+\frac{d\phi}{dz}\frac{d\psi}{dz}\right)dx\,dy\,dz$$

$$=\iint\phi\frac{d\psi}{dn}dS-\iiint\phi\nabla^2\psi\,dx\,dy\,dz\,\ldots\ldots(1)$$

$$=\iint\psi\frac{d\phi}{dn}dS-\iiint\psi\nabla^2\phi\,dx\,dy\,dz\,\ldots\ldots(2),$$

where the triple integrals extend throughout the volume of S, and the surface integrals over the surface of S, and dn denotes an element of the normal to S drawn outwards.

Integrating the left-hand side by parts, we obtain

$$\iiint\frac{d\phi}{dx}\frac{d\psi}{dx}dx\,dy\,dz=\left[\iint\phi\frac{d\psi}{dx}dy\,dz\right]-\iiint\phi\frac{d^2\psi}{dx^2}dx\,dy\,dz\,\ldots(3),$$

where the brackets denote that the double integral is to be taken between proper limits. Now since the surface is a closed surface, any line parallel to x, which enters the surface a given number of times, must issue from it the same number of times; also the x-direction cosine of the normal at the point of entrance will be of contrary sign to the same direction cosine at the corresponding point of exit; hence the surface integral

$$=\iint\phi\frac{d\psi}{dx}l\,dS.$$

Treating each of the other terms in a similar manner, we find that the left-hand side of (3)

$$=\iint \phi \frac{d\psi}{dn} dS - \iiint \phi \nabla^2 \psi \, dx\,dy\,dz.$$

The second equation (2) is obtained by interchanging ϕ and ψ.

59. We may deduce several important corollaries.

(i) Let $\psi = 1$, and let ϕ be the velocity potential of a liquid; then $\nabla^2\phi = 0$, and we obtain

$$0 = \iiint \nabla^2\phi \, dx\,dy\,dz = \iint \frac{d\phi}{dn} dS \quad \text{..............(4)}.$$

The right-hand side is the analytical expression for the fact that the total flux across the closed surface is zero; in other words as much liquid enters the surface as issues from it.

(ii) Let ϕ and ψ be both velocity potentials, then

$$\iint \phi \frac{d\psi}{dn} dS = \iint \psi \frac{d\phi}{dn} dS \quad \text{.................(5)}.$$

(iii) Let $\phi = \psi$, where ϕ is the velocity potential of a liquid; then

$$\iiint \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right\} dx\,dy\,dz = \iint \phi \frac{d\phi}{dn} dS \text{...(6)}.$$

If we multiply both sides of (6) by $\frac{1}{2}\rho$, the left-hand side is equal to the kinetic energy of the liquid; and the equation shows that the kinetic energy of a liquid whose motion is acyclic and irrotational, which is contained within a closed surface, depends solely upon the motion of the surface.

60. Let us now suppose that liquid contained within such a surface is originally at rest, and let the liquid be set in motion by means of an impulsive pressure p applied to every point of the surface. The motion produced must be necessarily irrotational, and acyclic; also if ϕ be its velocity potential, it follows from § 25 (39) that $p = -\rho\phi$. Now by the first proposition of § 57, the work done by the impulse is equal to

$$-\tfrac{1}{2}\iint p \frac{d\phi}{dn} dS = \tfrac{1}{2}\rho \iint \phi \frac{d\phi}{dn} dS;$$

whence equation (6) shows that the work done by the impulse is equal to the kinetic energy of the motion produced by it, as ought to be the case.

61. Let us in the next place suppose that liquid is contained within a closed surface which is in motion; and let the motion of the liquid be irrotational and acyclic; also let the surface be suddenly reduced to rest. Then if ϕ be the new velocity potential, $d\phi/dn = 0$, and therefore

$$\iiint \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right\} dxdydz = 0,$$

whence $d\phi/dx$, $d\phi/dy$, and $d\phi/dz$ are each zero, and therefore the liquid is reduced to rest.

62. In proving Green's Theorem, we have supposed that the region through which we integrate, is contained within a single closed surface, but if the region were bounded externally and internally by two or more closed surfaces, the theorem would still be true, provided we take the surface integral with the positive sign over the external boundary, and with the negative sign over each of the internal boundaries.

63. Let us suppose that the liquid is bounded internally by one or more closed surfaces S_1, S_2 &c., and externally by a very large fixed sphere whose centre is the origin. If T be the kinetic energy of the liquid,

$$T = \tfrac{1}{2}\rho \iint \phi \frac{d\phi}{dn}\, dS - \tfrac{1}{2}\rho \left[\iint \phi \frac{d\phi}{dn}\, dS\right],$$

where the square brackets indicate that the integral is to be taken over each of the internal boundaries.

If the liquid be at rest at infinity, the value of ϕ at S cannot contain any term of lower order than m/r, where m is a constant, whence

$$d\phi/dn = d\phi/dr = -m/r^2;$$

also if $d\Omega$ be the solid angle subtended by dS at the origin, $dS = r^2 d\Omega$; therefore

$$\iint \phi \frac{d\phi}{dn}\, dS = -\frac{m^2}{r} \iint d\Omega = -\frac{4\pi m^2}{r},$$

which vanishes when $r = \infty$. Hence the kinetic energy of an infinite liquid bounded internally by closed surfaces is

$$T = -\tfrac{1}{2}\rho \left[\iint \phi \frac{d\phi}{dn}\, dS\right] \quad \text{..................(7)},$$

where the surface integral is to be taken over each of the internal boundaries.

The preceding expression for the kinetic energy shows that, if the motion is acyclic and the *internal* boundaries of the liquid be suddenly reduced to rest, the whole liquid will be reduced to rest.

64. When a single solid is moving in an infinite liquid, the velocity potential must satisfy the following conditions;

(i) ϕ must be a single valued function, which at all points of the liquid satisfies the equation $\nabla^2\phi = 0$.

(ii) ϕ and its first derivatives must be finite and continuous at all points of the liquid, and must vanish at infinity, if any portion of the liquid extends to infinity.

(iii) At all points of the liquid which are in contact with a moving solid, $d\phi/dn$ must be equal to the normal velocity of the solid, where dn is an element of the normal to the solid drawn outwards; if any portion of the liquid is in contact with fixed boundaries, $d\phi/dn$ must be zero at every point of these fixed boundaries.

The most general possible motion of a solid may be resolved into three component velocities parallel to three rectangular axes (which may either be fixed or in motion), together with three angular velocities about these axes.

Let us therefore refer the motion to three rectangular axes Ox, Oy, Oz *fixed* in the solid, and let ϕ_1 be the velocity potential when the solid is moving with unit velocity parallel to Ox, and let χ_1 be the velocity potential when the solid is rotating with unit angular velocity about Ox. Let ϕ_2, ϕ_3, χ_2, χ_3 be similar quantities with respect to Oy and Oz. Also let u, v, w be the linear velocities of the solid parallel to, and ω_1, ω_2, ω_3 be its angular velocities about the axes.

We can now show that the velocity potential of the whole motion will be

$$\phi = u\phi_1 + v\phi_2 + w\phi_3 + \omega_1\chi_1 + \omega_2\chi_2 + \omega_3\chi_3 \quad \text{........(8)}.$$

For if λ, μ, ν be the direction cosines of the normal at any point x, y, z on the surface of the solid, we must have at the surface

$$\frac{d\phi_1}{dn} = \lambda, \quad \frac{d\phi_2}{dn} = \mu, \quad \frac{d\phi_3}{dn} = \nu,$$

$$\frac{d\chi_1}{dn} = \nu y - \mu z, \quad \frac{d\chi_2}{dn} = \lambda z - \nu x, \quad \frac{d\chi_3}{dn} = \mu x - \lambda y.$$

Hence $\dfrac{d\phi}{dn} = (u - y\omega_3 + z\omega_2)\lambda + (v - z\omega_1 + x\omega_3)\mu + (w - x\omega_2 + y\omega_1)\nu$

$=$ normal velocity of the solid.

65. If we substitute the value of ϕ from (8) in (7), it follows that T is a homogeneous quadratic function of the six velocities $u, v, w, \omega_1, \omega_2, \omega_3$, and therefore contains twenty-one terms. If we choose as our axes Ox, Oy, Oz, the principal axes at the centre of inertia O of the solid, the kinetic energy of the latter will be equal to

$$\tfrac{1}{2}M(u^2 + v^2 + w^2) + \tfrac{1}{2}(A_1\omega_1^2 + B_1\omega_2^2 + C_1\omega_3^2)$$

where M is the mass of the solid, and A_1, B_1, C_1 are its principal moments of inertia. Hence the kinetic energy T of the system, being the sum of the kinetic energies of the solid and liquid, is determined by the equation,

$$\begin{aligned} 2T = {} & Pu^2 + Qv^2 + Rw^2 + 2P'vw + 2Q'wu + 2R'uv \\ & + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + 2A'\omega_2\omega_3 + 2B'\omega_3\omega_1 + 2C'\omega_1\omega_2 \\ & + 2\omega_1(Lu + Mv + Nw) \\ & + 2\omega_2(L'u + M'v + N'w) \\ & + 2\omega_3(L''u + M''v + N''w) \quad \text{.................................(9).} \end{aligned}$$

The coefficients of the velocities in the preceding expression for the kinetic energy, are called *coefficients of inertia.* The quantities P, Q, R are called the *effective inertias of the solid parallel to the principal axes*, and the quantities A, B, C are called the *effective moments of inertia about the principal axes.* If the liquid extend to infinity, and there is only one moving solid, the coefficients of inertia depend solely upon the form and density of the solid and the density of the liquid, and not upon the coordinates which determine the position of the solid in space.

The values of these coefficients are

$$\left.\begin{aligned} P &= M - \rho\iint \phi_1 \frac{d\phi_1}{dn}\,dS = M - \rho\iint \phi_1\lambda\,dS \\ P' &= -\tfrac{1}{2}\rho\iint \phi_2\frac{d\phi_3}{dn}\,dS - \tfrac{1}{2}\rho\iint \phi_3\frac{d\phi_2}{dn}\,dS = -\rho\iint \phi_2\frac{d\phi_3}{dn}\,dS \end{aligned}\right\}\text{...(10)}$$

by (5); with similar expressions for the other coefficients.

When the form of the solid resembles that of an ellipsoid, which is symmetrical with respect to three perpendicular planes through its centre of inertia, and the motion is referred to the

principal axes of the solid at that point, the kinetic energy must remain unchanged when the direction of any one of the component velocities is reversed; hence the kinetic energy cannot contain any of the products of the velocities, and must therefore be of the form;

$$2T = Pu^2 + Qv^2 + Rw^2 + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \text{........} (11).$$

If in addition, the solid is one of revolution about the axis of z, the kinetic energy will not be altered if u is changed into v, and ω_1 into ω_2; whence $P = Q$, $A = B$, and

$$2T = P(u^2 + v^2) + Rw^2 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2 \text{......} (12).$$

Although every solid of revolution must be symmetrical with respect to all planes through its axis, it is not necessarily symmetrical with respect to a plane perpendicular to its axis. The solid formed by the revolution of a cardioid about its axis is an example of such a solid. In this case the kinetic energy will be unaltered when the signs of u, v or ω_3 are changed, and also when u is changed into v, and ω_1 into ω_2; hence in this case

$$2T = P(u^2 + v^2) + Rw^2 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2 + 2Nw(\omega_1 + \omega_2) \text{...} (13).$$

If the solid moves with its axis in one plane, (say zx), v and ω_1 must be zero, and the last term may be got rid of, by moving the origin to a point on the axis of z, whose distance from the origin is $-N/R$. This point is called the *Centre of Reaction*.

66. We must now find expressions for the component linear and angular momenta.

Since we are confining our attention to acyclic irrotational motion, it follows from § 59 that the motion of the liquid at any instant depends solely upon the motion of the surface of the solid; hence the motion which actually exists at any particular epoch, could be produced instantaneously from rest, by the application of suitable impulsive forces to the solid; and since impulsive forces are measured by the momenta which they produce, it follows that the resultant impulse which must be applied to the solid, must be equal to the resultant momentum of the solid and liquid.

Let ξ, η, ζ be the components parallel to the principal axes of the solid, of the impulsive force which must be applied to the solid, in order to produce the actual motion which exists at time t; and let λ, μ, ν be the components about these axes, of the impulsive couple. Then ξ, η, ζ are the components of the linear

momentum, and λ, μ, ν of the angular momentum of the solid and liquid.

Let p denote the impulsive pressure of the liquid, and let us consider the effect produced upon the solid by the application of the impulse whose components are ξ, η, ζ, λ, μ, ν.

By the ordinary equations of impulsive motion

$$Mu = \xi - \iint pl\,dS,$$

where l, m, n are the direction cosines of the normal at any point of S.

But if ϕ be the velocity potential of the motion instantaneously generated by the impulse, which is equal to the velocity potential which actually exists at time t, it follows from § 25, that $p = -\rho\phi$, whence

$$\begin{aligned}\xi &= Mu - \rho \iint \phi l\,dS \\ &= Mu - \rho \iint \phi \frac{d\phi_1}{dn}\,dS,\end{aligned}$$

since at the surface of the solid

$$l = d\phi_1/dn.$$

If $\mathfrak{T}'$, $\mathfrak{T}$ be the kinetic energies of the solid and liquid, it follows from (7) and (8) that

$$\begin{aligned}\frac{d\mathfrak{T}}{du} &= -\tfrac{1}{2}\rho \iint \phi \frac{d\phi_1}{dn}\,dS - \tfrac{1}{2}\rho \iint \phi_1 \frac{d\phi}{dn}\,dS \\ &= -\rho \iint \phi \frac{d\phi_1}{dn}\,dS,\end{aligned}$$

since by Green's Theorem both the double integrals are equal. Also

$$\frac{d\mathfrak{T}'}{du} = Mu,$$

whence

$$\begin{aligned}\xi &= \frac{d\mathfrak{T}'}{du} + \frac{d\mathfrak{T}}{du} \\ &= \frac{dT}{du}.\end{aligned}$$

We therefore see that the component momentum along the axis of x, is equal to the differential coefficient of the kinetic energy, with respect to the component velocity of the solid along the same axis; and by precisely similar reasoning, it can be shown that the component angular momentum about the axis of x, is

equal to the differential coefficient of the kinetic energy, with respect to the component angular velocity of the solid about this axis. We thus obtain the following equations for determining the momenta, viz.:

$$\left.\begin{aligned} \xi &= \frac{dT}{du}, & \eta &= \frac{dT}{dv}, & \zeta &= \frac{dT}{dw} \\ \lambda &= \frac{dT}{d\omega_1}, & \mu &= \frac{dT}{d\omega_2}, & \nu &= \frac{dT}{d\omega_3} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(14).$$

Equations (14) are well-known dynamical equations.

67. The preceding expressions for the kinetic energy and momenta, have been obtained in a direct manner, by means of hydrodynamical principles; but the reader who desires a short cut to equations (9) and (14), may begin by assuming the theorem, that the kinetic energy of a dynamical system is a homogeneous quadratic function of the velocities of the system. Since the kinetic energy of a liquid which surrounds a single moving solid, and whose motion is acyclic and irrotational, must vanish when the solid is reduced to rest, the kinetic energy of the liquid must be a homogeneous quadratic function of the velocities of the *solid.* This leads to (9). Also since the kinetic energy of an *infinite* liquid, when expressed in the form (9), cannot depend upon the position of the solid in space, the coefficients of the velocities must be constant quantities.

If ξ, η, ζ, λ, μ, ν be the component impulses, which must be applied to the solid, in order to generate from rest the motion which actually exists at time t, it follows from the first proposition of § 57, that

$$2T = \xi u + \eta v + \zeta w + \lambda\omega_1 + \mu\omega_2 + \nu\omega_3.$$

But since T is a homogeneous quadratic function of the velocities,

$$2T = u\frac{dT}{du} + v\frac{dT}{dv} + w\frac{dT}{dw} + \omega_1\frac{dT}{d\omega_1} + \omega_2\frac{dT}{d\omega_2} + \omega_3\frac{dT}{d\omega_3}.$$

Comparing these two equations, we obtain (14).

We are now in a position to solve a variety of problems connected with the motion of a single solid in an infinite liquid.

Motion of a Sphere.

68. Let us suppose that the centre of the sphere describes a plane curve, and let u, v be its component velocities parallel to the axes of x and y. Since every diameter of a sphere is a principal axis, the axes of x and y may be supposed to be fixed in direction, whence

$$\phi = -\frac{a^3}{2r^3}(ux + vy);$$

and since on account of symmetry $P = Q$, we have

$$T = \tfrac{1}{2}P(u^2 + v^2),$$

where
$$\begin{aligned} P &= M - \rho \iint \phi_1 l dS \\ &= M + \pi\rho a^3 \int_0^\pi \cos^2\theta \sin\theta \, d\theta \\ &= M + \tfrac{1}{2}M', \end{aligned}$$

where M' is the mass of the liquid displaced. Whence

$$T = \tfrac{1}{2}(M + \tfrac{1}{2}M')(u^2 + v^2),$$

and therefore

$$\xi = (M + \tfrac{1}{2}M')\,u, \quad \eta = (M + \tfrac{1}{2}M')\,v.$$

Let us now suppose that the sphere is descending under the action of gravity, and that the axis of y is drawn vertically downwards; we shall also suppose that the sphere is initially projected with a velocity, whose horizontal and vertical components are U and V.

Since the momentum parallel to x is constant throughout the motion,

$$(M + \tfrac{1}{2}M')\,u = \text{const.} = (M + \tfrac{1}{2}M')\,U,$$

whence
$$u = U.$$

To determine the force acting on the system, draw two horizontal planes above and below the sphere and at a considerable distance from it. Then the force on the whole system due to gravity is

$$Mg + \iint(p_1 - p_2 + g\rho z)\,dA - Mg' + h,$$

when p_1, p_2 are the hydrostatic pressures on the upper and lower planes, z the distance between them, and h that portion of the pressure due to the motion of the liquid. The integral vanishes; also h vanishes when the planes are at an infinite distance from

the sphere; whence the force is equal to $(M - M')g$. The equation giving the vertical motion is therefore

$$\frac{d\eta}{dt} = (M - M')g,$$

or

$$(M + \tfrac{1}{2}M')\frac{dv}{dt} = (M - M')g.$$

Whence if σ be the density of the sphere,

$$\frac{dv}{dt} = \frac{\sigma - \rho}{\sigma + \frac{1}{2}\rho}g,$$

and the sphere will describe a parabola with vertical acceleration $g(\sigma - \rho)/(\sigma + \frac{1}{2}\rho)$, in accordance with our previous result.

The motion of a right circular cylinder can be investigated in a precisely similar manner.

Motion of an Elliptic Cylinder.

69. Let u, v be the velocities of the cylinder parallel to the major and minor axes of its cross section, ω its angular velocity about its axis. On account of symmetry none of the products can appear, and therefore

$$T = \tfrac{1}{2}(Pu^2 + Qv^2 + A\omega^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

where P, Q, A are constant quantities.

Let us now suppose that no forces are in action, and that the solid and liquid are initially at rest; and let the cylinder be set in motion by means of an impulsive force F, whose line of action passes through its axis, and an impulsive couple which produces an initial angular velocity Ω.

Let us refer the motion to two fixed rectangular axes x and y, the former of which coincides with the direction of F, and let θ be the angle which the major axis of the cross section makes with the axis of x at time t.

Resolving the momenta along the axes of x and y, we obtain

$$\xi\cos\theta - \eta\sin\theta = F$$

$$\xi\sin\theta + \eta\cos\theta = 0,$$

whence since

$$\xi = Pu, \quad \eta = Qv,$$

we obtain

$$Pu = F\cos\theta, \quad Qv = -F\sin\theta \quad\ldots\ldots\ldots\ldots\ldots(16).$$

Since the kinetic energy remains constant throughout the motion, it follows that if we substitute the values of u, v from (16) in (15), and put β for the initial value of θ, we shall obtain

$$F^2\left(\frac{\cos^2\theta}{P}+\frac{\sin^2\theta}{Q}\right)+A\dot{\theta}^2=F^2\left(\frac{\cos^2\beta}{P}+\frac{\sin^2\beta}{Q}\right)+A\Omega^2,$$

or
$$A\dot{\theta}^2=A\Omega^2+F^2\left(\frac{1}{P}-\frac{1}{Q}\right)(\sin^2\theta-\sin^2\beta) \;\ldots\ldots\; (17).$$

We shall presently show that $Q>P$; it therefore follows that if

$$\Omega<F\sin\beta\sqrt{\frac{Q-P}{APQ}},$$

$\dot{\theta}$ will vanish, and the cylinder will oscillate; but if

$$\Omega>F\sin\beta\sqrt{\frac{Q-P}{APQ}},$$

$\dot{\theta}$ will never vanish, and the cylinder will make a complete revolution.

The integration of equation (17) requires elliptic functions, but without introducing these quantities, we can easily ascertain the character of the motion of the centre of inertia of the cylinder.

Let (x, y) be the coordinates of the centre of inertia referred to the fixed axes of x and y; then

$$\dot{x}=u\cos\theta-v\sin\theta,\quad \dot{y}=u\sin\theta+v\cos\theta \;\ldots\ldots\; (18).$$

Substituting the values of u, v from (16) we obtain

$$\dot{x}=\frac{F}{Q}+F\left(\frac{1}{P}-\frac{1}{Q}\right)\cos^2\theta,$$

$$\dot{y}=F\left(\frac{1}{P}-\frac{1}{Q}\right)\sin\theta\cos\theta.$$

These equations show that the centre of inertia of the cross section of the cylinder, moves along a straight line parallel to the direction of F with a uniform velocity F/Q, superimposed upon which is a variable periodic velocity, and that at the same time it vibrates perpendicularly to this line. This kind of motion frequently occurs in hydrodynamics, and a body moving in such a manner is called a *Quadrantal Pendulum*.

If
$$A\Omega^2=F^2\left(\frac{1}{P}-\frac{1}{Q}\right)\sin^2\beta,$$

which is the limiting case between oscillation and rotation, the

equations of motion admit of complete integration. Putting

$$I^2 = \frac{F^2}{A}\left(\frac{1}{P} - \frac{1}{Q}\right),$$

(17) becomes $\qquad \dot{\theta} = I \sin\theta$

whence $\qquad It = \log\tan\tfrac{1}{2}\theta.$

Therefore $\qquad \frac{dy}{d\theta} = \frac{IA}{F}\cos\theta,$

$$y = \frac{IA}{F}\sin\theta,$$

$$\frac{dx}{d\theta} = \frac{F}{PI}\operatorname{cosec}\theta - \frac{IA}{F}\sin\theta,$$

$$x = \frac{F}{PI}\log\tan\tfrac{1}{2}\theta + \frac{IA}{F}\cos\theta.$$

Putting $IA/F = c$, and eliminating θ we obtain the equation of the path, viz.

$$x = \frac{F}{PI}\log\frac{y}{c + \sqrt{c^2 - y^2}} + \sqrt{c^2 - y^2}.$$

The curves described by the centre of inertia of the cylinder in the three cases have been traced by Prof. Greenhill, and are shown in Figures 1, 2, 3 of the accompanying diagram.

70. We shall now show that for an elliptic cylinder $Q > P$.

When the cylinder is moving with unit velocity parallel to x, we have shown in § 48 that

$$\psi_x = c\epsilon^{-\eta+\beta}\sinh\beta\sin\xi,$$

and therefore $\qquad \phi_1 = -c\epsilon^{-\eta+\beta}\sinh\beta\cos\xi.$

Now at the surface

$$-\int\phi_1 l\,ds = c\sinh\beta\int\cos\xi\,dy$$

$$= c^2\sinh^2\beta\int_0^{2\pi}\cos^2\xi\,d\xi$$

$$= \pi b^2,$$

whence $\qquad P = M - \rho\int\phi_1 l\,ds$

$$= M\left(1 + \frac{\rho b}{\sigma a}\right),$$

where σ is the density of the cylinder.

The value of Q is evidently obtained by interchanging a and b, whence $Q > P$.

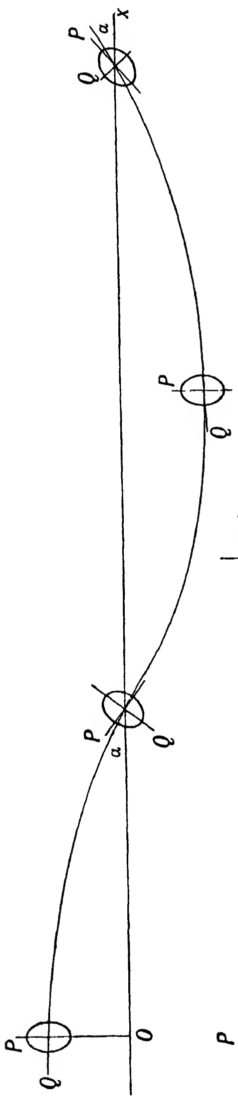


Fig. 1.

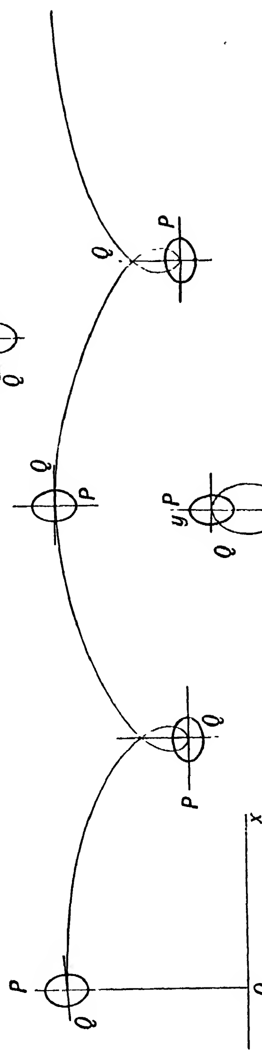


Fig. 2.

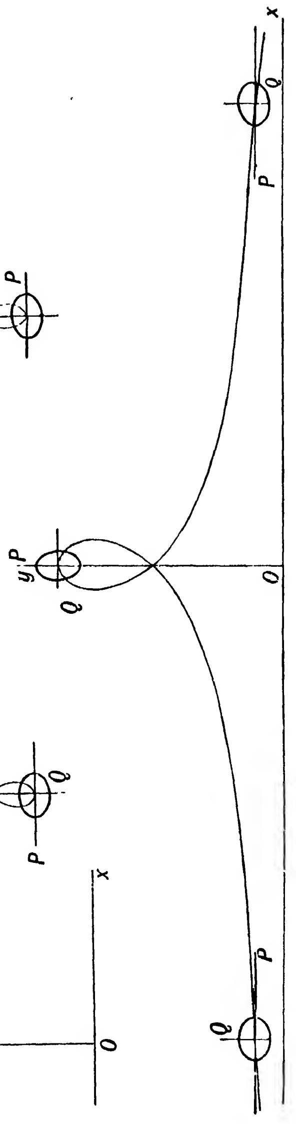


Fig. 3.

Similar results can be proved to be true in the case of an ellipsoid; from which it is inferred *that when any solid body is moving in an infinite liquid, the effective inertias corresponding to the greatest, mean, and least principal axes, are in descending order of magnitude.*

71. If the cylinder be projected parallel to a principal axis without rotation, it will continue to move in a straight line with uniform velocity; but if the direction of projection is not a principal axis, it will begin to rotate, and its angular velocity at any subsequent time will be determined by putting $\Omega = 0$ in (17). We shall now show that if the cylinder be projected parallel to a principal axis, its motion will be stable or unstable according as the direction of projection coincides with the minor or major axis.

Let us first suppose the cylinder projected parallel to its major axis, and that a slight disturbance is communicated to it. The equation determining the angular velocity is obtained by putting $\Omega = \beta = 0$ in (17); whence

$$A\dot{\theta}^2 = F^2\left(\frac{1}{P} - \frac{1}{Q}\right)\sin^2\theta,$$

and therefore differentiating, and remembering that in the beginning of the disturbed motion θ is a small quantity, we obtain

$$A\ddot{\theta} + F^2\left(\frac{1}{Q} - \frac{1}{P}\right)\theta = 0.$$

Since $Q > P$, the coefficient of θ is negative, which shows that the motion is unstable.

If the cylinder is projected parallel to its minor axis we must put $\beta = \frac{1}{2}\pi$; also if $\chi = \frac{1}{2}\pi - \theta$, χ will be a small quantity in the beginning of the disturbed motion; whence (17) becomes

$$A\dot{\chi}^2 = -F^2\left(\frac{1}{P} - \frac{1}{Q}\right)\sin^2\chi,$$

whence

$$A\ddot{\chi} + F^2\left(\frac{1}{P} - \frac{1}{Q}\right)\chi = 0.$$

Since the coefficient of χ is positive, the motion is stable.

It can also be shown that if an ellipsoid be projected parallel to a principal axis, without rotation, the motion will be unstable unless the direction of projection coincides with the *least axis.*

We shall however presently show, that if an ovary ellipsoid be projected parallel to its axis, the motion will be stable, provided a sufficiently large angular velocity be communicated to the solid about its axis.

72. Let us now investigate the motion of an elliptic cylinder, which descends from rest under the action of gravity.

Let the axis of y be horizontal, and the axis of x be drawn vertically downwards.

The equations of momentum are

$$\xi \sin\theta + \eta\cos\theta = 0,$$

$$\frac{d}{dt}(\xi\cos\theta - \eta\sin\theta) = (M - M')\,g,$$

from the last of which we obtain

$$\xi\cos\theta - \eta\sin\theta = (M - M')\,gt.$$

Solving these equations and recollecting that $\xi = Pu$, $\eta = Qv$, we obtain

$$Pu = (M - M')\,gt\cos\theta, \quad Qv = -(M - M')\,gt\sin\theta.$$

Substituting these values of u and v in (18), we obtain

$$\left.\begin{aligned} \dot{x} &= \left(\frac{\cos^2\theta}{P} + \frac{\sin^2\theta}{Q}\right)(M - M')\,gt \\ \dot{y} &= \left(\frac{1}{P} - \frac{1}{Q}\right) gt\sin\theta\cos\theta \end{aligned}\right\}\ldots\ldots\ldots\ldots(19).$$

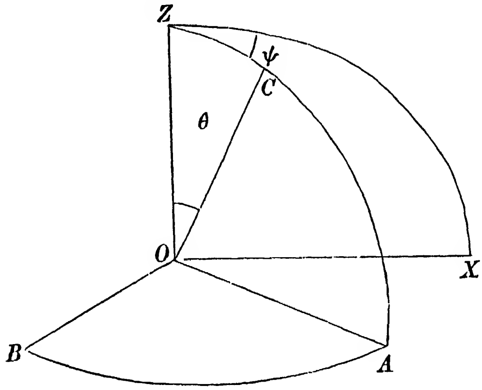
The equation of energy gives

$$\left(\frac{\cos^2\theta}{P} + \frac{\sin^2\theta}{Q}\right)(M - M')^2 g^2t^2 + A\dot{\theta}^2 = 2(M - M')\,gx.$$

If we differentiate this equation with respect to t, we can eliminate $\dot{x}$ by means of (19); but the resulting equation would be difficult to deal with. We see however from the first of (19), that $\dot{x}$ is always positive, and therefore the cylinder moves downwards with a variable velocity, which depends upon the inclination of its major axis to the vertical, as well as upon the time. We also see from the second equation, that the horizontal velocity vanishes, whenever the major axis becomes horizontal or vertical; but if the motion should be of such a character that θ always lies between 0 and $\frac{1}{2}\pi$, the horizontal velocity will never vanish.

Helicoidal Steady Motion of a Solid of Revolution.

73. In the figure let OC be the axis of the solid of revolution, O its centre of inertia, and let the solid be rotating with angular velocity Ω about its axis.



Let the solid be set in motion by means of an impulsive force F along OZ, and an impulsive couple G about OZ, and let α be the angle which OC initially makes with OZ.

Let ψ be the angle which the plane ZOC makes at time t with a plane ZOX, which is parallel to some fixed plane, and let the former plane cut the equatorial plane in OA; also let $ZOC = \theta$.

It will be convenient to refer the motion to three moving axes, OA, OB, OC, where OB is the equatorial axis which is perpendicular to OA.

Resolving the linear momentum of the system along OZ, OX and a line OY perpendicular to the plane ZOX, we obtain

$$-\xi \sin\theta + \zeta \cos\theta = F,$$

$$(\xi \cos\theta + \zeta \sin\theta)\cos\psi - \eta \sin\psi = 0,$$

$$(\xi \cos\theta + \zeta \sin\theta)\sin\psi + \eta \cos\psi = 0,$$

whence $$\xi = -F\sin\theta, \quad \eta = 0, \quad \zeta = F\cos\theta \;\ldots\ldots\ldots\; (20).$$

Since the components of momentum parallel to the axes of X and Y (which are fixed in direction, but not in position because O is in motion) are zero throughout the motion, the angular

momentum about OZ is constant[1], whence

$$-A\omega_1 \sin\theta + C\Omega\cos\theta = G + C\Omega\cos\alpha \quad\text{.........(21).}$$

The equation of energy gives

$$Pu^2 + Rw^2 + A(\omega_1^2 + \dot\theta^2) = \text{const.} \quad\text{............(22),}$$

also

$$\xi = Pu, \quad \zeta = Rw,$$

and therefore by (20) and (21) this becomes

$$F^2\left(\frac{\sin^2\theta}{P} + \frac{\cos^2\theta}{R}\right) + \frac{\{G + C\Omega(\cos\alpha - \cos\theta)\}^2}{A\sin^2\theta} + A\dot\theta^2 = \text{const.}$$

$$= \text{its initial value} \quad\text{...... (23).}$$

This equation determines the inclination θ of the axis.

So far our equations have been perfectly general, we shall now introduce the conditions of steady motion. These are

$$\theta = \alpha, \quad \dot\psi = \mu, \quad \ddot\theta = \dot\theta = 0 \quad\text{.............. (24).}$$

Now $\omega_1 = -\dot\psi\sin\alpha = -\mu\sin\alpha$, whence (21) becomes

$$A\mu\sin^2\alpha = G \quad\text{.......................(25).}$$

Differentiating (23) with respect to t and using (24) and (25) we obtain

$$A\mu^2\cos\alpha - C\Omega\mu + \left(\frac{1}{R} - \frac{1}{P}\right)F^2\cos\alpha = 0 \quad\text{.........(26).}$$

This is a quadratic equation for determining μ, when Ω and F are given. Now μ must necessarily be a real quantity, and therefore the condition that steady motion may be possible is that

$$C^2\Omega^2 > 4F^2A\cos^2\alpha\left(\frac{1}{R} - \frac{1}{P}\right) \quad\text{..............(27),}$$

and since

$$Rw = \zeta = F\cos\alpha,$$

the condition becomes

$$C^2\Omega^2 > 4AR^2w^2\left(\frac{1}{R} - \frac{1}{P}\right) \quad\text{...............(28).}$$

[1] It was shown by Hayward (*Trans. Camb. Phil. Soc.* Vol. x.) that when the origin as well as the axes are in motion, the Principle of Angular Momentum is expressed by three equations of the form

$$\frac{d\nu'}{dt} - v\xi' + u\eta' - \lambda'\theta_2 + \mu'\theta_1 = N.$$

Since the direction of the axes of X, Y and Z are fixed, $\theta_1 = \theta_2 = 0$; also since the momenta ξ' and η' parallel to X and Y are zero, the equation reduces to $d\nu'/dt = N$; which gives $\nu = \text{const.}$, when $N = 0$. This result can of course be proved by elementary methods.

If the solid of revolution is oblate (such as a planetary ellipsoid) $R > P$, and therefore (28) is always satisfied; but if the solid is prolate (such as an ovary ellipsoid) $P > R$, and therefore steady motion will not be possible unless Ω exceeds a certain value.

In order to find the path described by the centre of inertia of the solid in steady motion, we have, since $\psi = \mu t$,

$$\dot{x} = (u \cos\alpha + w \sin\alpha)\cos\psi = F\left(\frac{1}{R} - \frac{1}{P}\right)\sin\alpha \cos\alpha \cos\mu t,$$

$$\dot{y} = (u \cos\alpha + w \sin\alpha)\sin\psi = F\left(\frac{1}{R} - \frac{1}{P}\right)\sin\alpha \cos\alpha \sin\mu t,$$

$$\dot{z} = w\cos\alpha - u\sin\alpha \qquad = F\left(\frac{\sin^2\alpha}{P} + \frac{\cos^2\alpha}{R}\right),$$

which shows that the centre of inertia describes a helix.

74. In order to find whether the steady motion is stable or unstable, differentiate (23) with respect to t, and we obtain

$$A\ddot{\theta} + f(\theta) = 0 \quad\text{.......................}(29),$$

where

$$f(\theta) = \tfrac{1}{2}F^2\left(\frac{1}{P} - \frac{1}{R}\right)\sin 2\theta + \frac{C\Omega}{A\sin\theta}\{G + C\Omega(\cos\alpha - \cos\theta)\}$$

$$- \frac{\cos\theta}{A\sin^3\theta}\{G + C\Omega(\cos\alpha - \cos\theta)\}^2.$$

The condition for steady motion is, that $f(\alpha) = 0$, which leads to (26), whence writing $\theta = \alpha + \chi$ where χ is small, (29) becomes

$$A\ddot{\chi} + f'(\alpha)\chi = 0,$$

and the condition of stability requires that $f'(\alpha)$ should be positive. Now

$$f'(\alpha) = A\mu^2(1 + 2\cos^2\alpha) - 3C\Omega\mu\cos\alpha + \frac{C^2\Omega^2}{A} - F^2\left(\frac{1}{R} - \frac{1}{P}\right)\cos 2\alpha,$$

whence eliminating Ω by (26) this becomes

$$A^2\mu^2 f'(\alpha) = A^2\mu^4 + A\mu^2F^2\left(\frac{1}{R} - \frac{1}{P}\right)(1 - 3\cos^2\alpha) + F^4\left(\frac{1}{R} - \frac{1}{P}\right)^2\cos^2\alpha.$$

The condition that the right-hand side should be positive is that

$$A^2F^4\left(\frac{1}{R} - \frac{1}{P}\right)^2\sin^2\alpha\,(9\cos^2\alpha - 1) > 0,$$

which requires that α should lie between $\cos^{-1}\frac{1}{3}$ and 0, or between $\pi - \cos^{-1}\frac{1}{3}$ and π.

As a particular example let the solid be projected point foremost; then $\alpha = 0$ and $G = 0$, and therefore since θ is a small quantity in the beginning of the disturbed motion

$$f(\theta) = \left\{\frac{C^2\Omega^2}{4A} + F^2\left(\frac{1}{P} - \frac{1}{R}\right)\right\}\theta.$$

If therefore $R > P$ the motion is always stable, whether there is or is not rotation, and consequently the forward motion of a planetary ellipsoid is always stable; but if $P > R$, it follows that since $F = Rw$, the motion will be unstable unless

$$\Omega > \frac{2Rw}{C}\sqrt{A\left(\frac{1}{R} - \frac{1}{P}\right)}.$$

The motion of an ovary ellipsoid is therefore unstable, unless the ratio of its angular velocity to its forward velocity exceeds a certain value.

75. These results have an application in gunnery.

When an elongated body, such as a bullet, is fired from a gun with a high velocity, the effect of the air upon its motion cannot be neglected; and if the air is treated as an incompressible fluid, the previous investigation shows that the bullet will tend to present its flat side to the air, and also to deviate from its approximately parabolic path, unless it be endowed with a rapid rotation about its axis. Hence the bores of all guns destined for long ranges are rifled, by means of which a rapid rotation is communicated to the bullet before it leaves the barrel. The effect of the rifling tends to keep the bullet moving point foremost, and to ensure its travelling along an approximately parabolic path in a vertical plane. Moreover when a bullet is moving with a high velocity, the effect of friction cannot be neglected; and it is obvious that when the bullet is moving with its flat side foremost, the effect of frictional resistance will be much greater than when it is moving point foremost, and therefore the bullet will not travel so far in the former as in the latter case. The hydrodynamical theory therefore explains the necessity of rifling guns.

Motion of a Cylinder parallel to a Plane.

76. We have thus far supposed that the liquid extends to infinity in all directions; we shall now suppose that the liquid

is bounded by a fixed plane, and shall enquire what effect the plane boundary produces on the motion of a circular cylinder.

Let the axis of y be drawn perpendicularly to the plane, and let the origin be in the plane, and let (x, y) be the coordinates of the centre of the cylinder, (u, v) its velocities parallel to and perpendicular to the plane.

The kinetic energy of the solid and liquid must be a homogeneous quadratic function of u and v, but since the kinetic energy is necessarily unchanged when the sign of u is reversed, the product uv cannot appear. We may therefore write

$$T = \tfrac{1}{2}(Ru^2 + R'v^2) \qquad (30).$$

The coefficients R, R' depend upon the distance of the cylinder from the plane, and are therefore functions of y but not of x; and as a matter of fact their values are equal. It will not however be necessary to assume the equality of R and R', since our object will be attained provided we can show that R and R' diminish as y increases.

In order to produce from rest the motion which actually exists at time t, we must apply to the cylinder impulsive forces whose components are X, Y; and we must also apply at every point of the plane boundary an impulsive pressure, which is just sufficient to prevent the liquid in contact with the plane from having any velocity perpendicular to the plane. The work done by the impulsive pressure is zero, whilst the work done by the impulses X, Y is

$$\tfrac{1}{2}(Xu + Yv) \qquad (31),$$

which must be equal to T. Now (30) may be written in the form

$$T = \tfrac{1}{2}\left(u\frac{dT}{du} + v\frac{dT}{dv}\right) \qquad (32),$$

whence comparing (31) and (32) we see that

$$X = \frac{dT}{du} = Ru, \quad Y = \frac{dT}{dv} = R'v \qquad (33),$$

and therefore

$$T = \tfrac{1}{2}\left(\frac{X^2}{R} + \frac{Y^2}{R'}\right) \qquad (34).$$

The last equation gives the kinetic energy in terms of the impulsive forces X, Y applied to the cylinder.

Let us now suppose that the cylinder instead of being at a distance y from the plane, is at a distance y_1, where $y_1 > y$; and

let R_1, R_1' be the values of R, R' at y_1. Then if the cylinder were set in motion by the same impulses, the work done would be

$$T_1 = \frac{1}{2}\left(\frac{X^2}{R_1} + \frac{Y^2}{R_1'}\right) \quad\text{......................(35).}$$

Now the effect of the plane boundary is to produce a constraint, and the effect of this constraint evidently diminishes as the distance of the cylinder from the plane increases, and therefore by Bertrand's theorem, $T_1 > T$. Hence

$$X^2\left(\frac{1}{R_1} - \frac{1}{R}\right) + Y^2\left(\frac{1}{R_1'} - \frac{1}{R'}\right)$$

is positive for all values of X and Y, which requires that

$$R > R_1, \quad R' > R_1';$$

hence R, R' diminish as y increases, and consequently their differential coefficients with respect to y are *negative.*

77. We can now determine the motion of the cylinder.

The momentum parallel to x is equal to dT/du and is constant; whence

$$Ru = \text{const.} = X \quad\text{......................(36).}$$

Since the kinetic energy is constant, we have

$$Ru^2 + R'v^2 = \text{const.} = 2T \quad\text{..................(37).}$$

Differentiating (37) with respect to t, and eliminating du/dt by (36), we obtain

$$\dot{v} = -\frac{1}{2R'}\left(v^2\frac{dR'}{dy} - u^2\frac{dR}{dy}\right) \quad\text{...............(38).}$$

From this equation we can ascertain the effect of the plane boundary; for if the cylinder is projected perpendicularly to the plane, $u = 0$, and

$$\dot{v} = -\frac{v^2}{2R'}\frac{dR'}{dy}.$$

Now dR'/dy is negative, and therefore $\dot{v}$ is positive; whence it follows that whether the cylinder be moving from or towards the plane, the force exerted by the liquid upon the cylinder will always be a *repulsion* from the plane, which is equal to

$$-\frac{Mv^2}{2R'}\frac{dR'}{dy}.$$

Hence if the cylinder be in contact with the plane, and a small

velocity perpendicular to the plane be communicated to it, the cylinder will begin to move away from the plane with gradually increasing velocity. This velocity cannot however increase indefinitely, for that would require the energy to become infinite, which is impossible, since the energy remains constant and equal to its initial value. If R_0' denote the value of R' when the cylinder is in contact with the plane, v_0 the initial velocity, and v the velocity when the cylinder is at an infinite distance from the plane, the equation of energy gives

$$R_0'v_0^2 = R_\infty'v^2.$$

The value of $R_\infty' = M + M'$, since the motion is the same as if the plane boundary did not exist, whence the ratio of the terminal to the initial velocity is

$$\frac{v}{v_0} = \sqrt{\frac{R_0'}{M+M'}}.$$

Let us now suppose that the cylinder is projected parallel to the plane with initial velocity u_0. By (38) the initial acceleration $\dot{v}_0$ perpendicular to the plane is

$$\dot{v}_0 = \frac{u_0^2}{2R'}\frac{dR}{dy};$$

and since dR/dy is negative, the cylinder will be attracted towards the plane, and will ultimately strike it.

78. All the results of the last two sections are true in the case of a sphere, and can be proved in the same manner. Moreover the motion will be unaltered, if we remove the plane boundary, and suppose that on the other side, an infinite liquid exists in which another equal cylinder or sphere is moving with velocities u, $-v$. The second cylinder or sphere is therefore the image of the first. Our results are therefore applicable to the case of two equal cylinders or spheres, which are moving with equal and opposite velocities along the line joining their centres; or to the case in which the cylinders or spheres are projected perpendicularly to the line joining their centres, with velocities which are equal and in the same direction.

These results have however a wider application, for according to the views of Faraday and Maxwell, the action which is observed to take place between electrified bodies is not due to any direct action which electrified bodies exert upon one another, but to

something which takes place in the dielectric medium surrounding these bodies; and although the preceding hydrodynamical results do not of course furnish any explanation of what takes place in dielectric media, they establish the fact that two bodies which are incapable of exerting any direct influence upon one another, are capable of producing an apparent attraction or repulsion upon one another, when they are in motion in a medium which may be treated as possessing the properties of an incompressible fluid.

EXAMPLES.

1. A light cylindrical shell whose cross section is an ellipse is filled with water, and placed at rest on a smooth horizontal plane in its position of unstable equilibrium. If it is slightly disturbed, prove that it will pass through its position of stable equilibrium with angular velocity ω, given by the equation

$$\omega^2 = \frac{8g(a^2+b^2)}{(a+b)^2(a-b)}.$$

2. An elliptic cylindrical shell, the mass of which may be neglected, is filled with water, and placed on a horizontal plane very nearly in the position of unstable equilibrium with its axis horizontal, and is then let go. When it passes through the position of stable equilibrium, find the angular velocity of the cylinder, ~~(i) when~~ the horizontal ~~plane is perfectly smooth~~, (ii) when it is perfectly rough; and prove that ~~in these~~ two cases, the squares of the angular velocities ~~are in the ratio~~

$$\cancel{(a^2-b^2)^2+4b^2(a^2+b^2) : (a^2-b^2)^2},$$

2*a* and 2*b* being the axes of the cross section of the cylinder.

3. A pendulum with an elliptic cylindrical cavity filled with liquid, the generating lines of the cylinder being parallel to the axis of suspension, performs finite oscillations under the action of gravity. If l be the length of the equivalent pendulum, and l' the length when the liquid is solidified, prove that

$$l' - l = \frac{ma^2b^2}{h(M+m)(a^2+b^2)},$$

where M is the mass of the pendulum, m that of the liquid, h the

distance of the centre of gravity of the whole mass from the axis of suspension, and a, b the semi-axes of the elliptic cavity.

4. Find the ratio of the kinetic energy of the infinite liquid surrounding an oblate spheroid, moving with given velocity in its equatorial plane, to the kinetic energy of the spheroid; and denoting this ratio by P, prove that if the spheroid swing as the bob of a pendulum under gravity, the distance between the axis of the suspension and the axis of the spheroid being c, the length of the simple equivalent pendulum is

$$\frac{(1+P)\,c + 2a^2/5c}{1-\rho/\sigma},$$

where a is the equatorial radius, σ and ρ the densities of the spheroid and liquid respectively.

5. A pendulum has a cavity excavated within it, and this cavity is filled with liquid. Prove that if any part of the liquid be solidified, the time of oscillation will be increased.

6. A closed vessel filled with liquid of density ρ, is moved in any manner about a fixed point O. If at any time the liquid were removed, and a pressure proportional to the velocity potential were applied at every point of the surface, the resultant couple due to the pressure would be of magnitude G, and its direction in a line OQ. Show that the kinetic energy of the liquid was proportional to $\frac{1}{2}\rho\omega G\cos\theta$, where ω is the angular velocity of the surface, and θ the angle between its direction and OQ.

7. Liquid is contained in a simply-connected surface S; if ϖ is the impulsive pressure at any point of the liquid due to any arbitrary deformation of S, subject to the condition that the enclosed volume is not changed, and ϖ' the impulsive pressure for a different deformation, show that

$$\iint \varpi \frac{d\varpi'}{dn}\,dS = \iint \varpi' \frac{d\varpi}{dn}\,dS.$$

8. If a sphere be immersed in a liquid, prove that the kinetic energy of the liquid due to a given deformation of its surface, will be greater when the sphere is fixed than when it is free.

CHAPTER IV.

WAVES.

79. BEFORE discussing the dynamical theory of waves, we shall commence by explaining what a wave is.

Let us suppose that the equation of the free surface of a liquid at time t, is

$$y = a \sin (mx - nt) \qquad (1),$$

where the axis of x is horizontal, the axis of y is measured vertically upwards, and a, m and n are constants.

The initial form of the free surface, i.e. its form when $t = 0$, is $y = a \sin mx$, which is the curve of sines. The maximum values of y occur when $mx = (2k + \frac{1}{2})\pi$, where k is zero or any positive or negative integer; and this maximum value is equal to a. The minimum values of y occur when $mx = (2k + \frac{3}{2})\pi$, where k is zero or any positive or negative integer; and this minimum value is equal to $-a$. As x increases from 0 to $\frac{1}{2}\pi/m$, y increases from 0 to a, and as x increases from $\frac{1}{2}\pi/m$ to π/m, y decreases from a to 0. As x increases from π/m to $\frac{3}{2}\pi/m$, y is negative; and when x has the latter value, y has attained its greatest negative value, which is equal to $-a$; as x increases from $\frac{3}{2}\pi/m$ to $2\pi/m$, y numerically decreases from $-a$ to 0.

The values of y comprised between

$$x = 2k\pi/m \text{ and } x = 2(k + 1)\pi/m,$$

evidently go through exactly the same cycle of changes.

When the motion of a liquid is such, that its free surface is represented by an equation such as (1), the motion is called *wave motion*.

The quantity a, which is equal to the maximum value of y, is called the *amplitude*; and the distance $2\pi/m$, between two consecutive maxima values of y, is called the *wave length.*

In order to ascertain the form of the free surface at time t, let us transfer the origin to a point $\xi = nt/m$; then if x' be the abscissa at time t referred to the new origin, of the point whose abscissa referred to the old origin is x, we have $x = x' + \xi$ and

$$y = a \sin (mx' + m\xi - nt) = a \sin mx'.$$

The form of the free surface at time t, is therefore obtained by making the point which initially coincided with the origin, travel along the axis of x, with velocity n/m.

The velocity of this point is called the *velocity of propagation* of the wave.

If V be the velocity of propagation, and λ the wave length, we thus obtain the equations

$$m = 2\pi/\lambda, \quad V = n/m \qquad (2),$$

and therefore (1) may be written

$$y = a \sin \frac{2\pi}{\lambda} (x - Vt) \qquad (3).$$

If n, and therefore V, were negative, equations (1) and (3) would represent a wave travelling in the opposite direction.

The position of the free surface at time t, is exactly the same as at time $t + 2k\pi/n$, or $t + 2k\lambda/V$, since $n = 2\pi V/\lambda$; the quantity λ/V is called the *periodic time*, or shortly the *period*, and is equal to the time which the crest of one wave occupies in travelling from its position at time t to the position occupied by the next crest at the same epoch. If τ denote the period, we evidently have

$$\lambda = V\tau \qquad (4),$$

and (1) may be written in the form

$$y = a \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{\tau}\right) \qquad (5),$$

which is a form sometimes convenient in Physical Optics.

From (4) we see that for waves travelling with the same velocity, the period increases with the wave length.

The reciprocal of the period, which is the number of vibrations executed per unit of time, is called the *frequency*. If therefore we

have a medium which propagates waves of all lengths with the same velocity, equation (4) shows that the number of vibrations executed in a second increases as the wave length diminishes. This remark is of importance in the Theory of Sound.

Let us now suppose that two waves are represented by the equations

$$y = a \sin \frac{2\pi}{\lambda}(x - Vt),$$

$$y' = a \sin \frac{2\pi}{\lambda}(x - Vt - e).$$

The amplitudes, wave lengths and velocities of propagation of the two waves are equal, but the second wave is in advance of the first; for if in the first equation we put $t + e/V$ for t, the two equations become identical. It therefore follows that the distance at which the second wave is in advance of the first is equal to e. The quantity e is called the *phase* of the wave.

80. Waves which are represented by equations such as (1), are called *progressive waves*; their wave lengths are equal to $2\pi/m$, and their velocities of propagation to n/m. If such waves are travelling along the surface of water under the action of gravity, they may be conceived to have been produced by communicating to the free surface an initial displacement $y = a \sin mx$, together with an initial velocity $-an \cos mx$. We therefore see that the wave length depends solely on the initial displacement, but that the velocity of propagation depends upon the initial velocity as well as upon the initial displacement.

If we combine the two waves, which are obtained by writing $\pm n$ for n in (1) and add the results, we shall obtain

$$\begin{aligned} y &= a \sin (mx - nt) + a \sin (mx + nt) \\ &= 2a \sin mx \cos nt \qquad (6). \end{aligned}$$

Such a wave is called a *stationary wave*. It is produced by means of an initial displacement alone, and gives the form of the free surface at time t, when the latter is displaced into the form of the curve $y = 2a \sin mx$, and is then left to itself.

In equations (1) or (6), m, which is equal to $2\pi/\lambda$, is always supposed to be given, and the problem we have to solve, consists in finding the value of n, which determines the velocity of propagation.

81. In most problems relating to small oscillations, the motion is supposed to be sufficiently slow for the quadratic terms which occur in the equations of motion to be neglected. Under these circumstances, the equations become linear and usually admit of a solution in which the time enters in the form of the factor $\epsilon^{\iota nt}$. Throughout the present chapter this hypothesis will be made; but it may be remarked that a solution of the *complete* equations of motion has been obtained by Gerstner, which leads to a species of trochoidal wave involving molecular rotation. The theory of waves involving molecular rotation is not of any great interest in the dynamics of a *frictionless* liquid; but it is of the highest importance in the dynamics of *actual* liquids, which are *viscous*, inasmuch as the motion of a viscous liquid always involves molecular rotation.

We shall now proceed to consider the irrotational motion of liquid waves in two dimensions, under the action of gravity.

The solution of the problem involves the determination of a velocity potential ϕ, which satisfies the following three conditions:

(i) ϕ must satisfy Laplace's equation, and together with its first derivatives, must be finite and continuous at every point of the liquid.

(ii) ϕ must satisfy the given boundary conditions at the fixed boundaries of the liquid.

(iii) ϕ must be determined, so that the free surface of the liquid is a surface of constant pressure.

To find the condition to be satisfied at the free surface, let the origin be taken in the undisturbed surface, and let the axis of x be measured in the direction of propagation of the waves, and let the axis of z be measured vertically upwards.

The pressure at any point of the liquid is determined by the equation

$$p/\rho + gz + \dot{\phi} + \tfrac{1}{2}q^2 = C \quad\text{.....................(7).}$$

The equation of the surfaces of constant pressure is $p =$ const., and since the free surface is included in this family of surfaces, *and must also satisfy the kinematical condition of a bounding surface*, it follows from § 12 (17) that

$$\frac{dp}{dt} + u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz} = 0 \quad\text{.....................(8).}$$

Substituting the value of p from (7) in (8), and neglecting squares and products of the velocity, we obtain

$$\frac{d^2\phi}{dt^2} + g\frac{d\phi}{dz} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

This is the condition to be satisfied at a free surface, where $z = 0$.

Waves in a Liquid of given Depth.

82. We shall now find the velocity of propagation of two-dimensional waves travelling in an ocean of depth h.

The equation of continuity is

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dz^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

The boundary condition at the bottom of the liquid is

$$\frac{d\phi}{dz} = 0, \text{ when } z = -h \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

To satisfy (10) assume

$$\phi = F(z)\cos(mx - nt) \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Substituting in (10) we obtain

$$\frac{d^2F}{dz^2} - m^2F = 0,$$

the solution of which is

$$F = P\cosh mz + Q\sinh mz;$$

whence $\qquad \phi = (P\cosh mz + Q\sinh mz)\cos(mx - nt).$

Substituting in (11) and (9) we obtain

$$P\sinh mh = Q\cosh mh,$$

$$Pn^2 = Qmg;$$

whence eliminating P and Q, and taking account of (2), we obtain

$$V^2 = (g\lambda/2\pi)\tanh(2\pi h/\lambda) \quad \ldots\ldots\ldots\ldots\ldots (13),$$

which determines the velocity of propagation.

If the lengths of the waves are large in comparison with the depth of the liquid, h/λ is small, and the preceding result becomes

$$V^2 = gh \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

which determines the velocity of propagation of long waves in shallow water.

If the depth of the liquid is large in comparison with the wave length, h/λ is large, and $\tanh 2\pi h/\lambda = 1$ approximately, whence

$$V^2 = g\lambda/2\pi \text{ .... } (15),$$

which determines the velocity of propagation of deep sea waves.

The last result may be obtained directly, for the value of F may be written in the form

$$F = A\epsilon^{mz} + B\epsilon^{-mz},$$

and since ϕ, and therefore F, cannot be infinite when $z = -\infty$, $B = 0$, and (9) at once gives the required result

83. Equation (15), which determines the velocity of propagation of deep-sea waves, may be written in the form $n^2 = mg$; whence the form of the free surface at time t is

$$y = a\cos(mx - m^{\frac{1}{2}}g^{\frac{1}{2}}t) \text{ .... } (16),$$

corresponding to the initial form $y = a\cos mx$. If the initial form were $y = F(x)$, Fourier's theorem enables us to express $F(x)$ in the form of a definite integral involving $\cos mx$; hence by (16) the form of the free surface at any subsequent time can be written down in terms of these quantities; but the difficulty of evaluating the definite integral is usually so great, that the results are rarely of much practical utility.

As an example of the use of definite integrals, let us suppose that the initial form of the free surface is

$$y = a\exp(\mp x/c),$$

where the upper and lower signs are to be taken on the positive and negative sides of the origin respectively. When x is positive

$$\exp(-x/c) = \frac{2}{\pi}\int_0^\infty \frac{\cos xu/c \,.\, du}{1+u^2}$$

$$= \frac{2c}{\pi}\int_0^\infty \frac{\cos mx\, dm}{1+m^2c^2},$$

whence the equation of the free surface at time t is

$$y = \frac{2ac}{\pi}\int_0^\infty \frac{\cos(mx - m^{\frac{1}{2}}g^{\frac{1}{2}}t)\, dm}{1+m^2c^2}$$

84. Returning to the general case, we see that ϕ is of the form

$$\phi = A\cosh m(z+h)\cos(mx - nt).$$

If η be the elevation of the free surface above the undisturbed surface, we must have

$$\dot{\eta} = d\phi/dz \text{ when } z = 0 \qquad (17),$$

whence substituting the value of ϕ in (17), and suitably choosing the origin we obtain

$$\eta = -Amn^{-1} \sinh mh \sin (mx - nt).$$

Let (x, z) be the coordinates of an element of liquid when undisturbed, (ξ, ζ) its horizontal and vertical displacements, also let $x' = x + \xi$, $z' = z + \zeta$; then

$$\dot{\xi} = d\phi/dx' = -Am \cosh m(z' + h) \sin (mx' - nt)$$

$$\dot{\zeta} = d\phi/dz' = \quad Am \sinh m(z' + h) \cos (mx' - nt).$$

Since the displacement is small we may put $x = x'$, $z = z'$ as a first approximation, and we obtain

$$\xi = -a \cosh m(z + h) \cos (mx - nt)$$

$$\zeta = -a \sinh m(z + h) \sin (mx - nt),$$

where $Am/n = a$; whence the elements of liquid describe the ellipse

$$\xi^2/\cosh^2 m(z + h) + \zeta^2/\sinh^2 m(z + h) = a^2.$$

When the depth of the liquid is very great we may put $h = \infty$, and the hyperbolic functions must be replaced by exponential ones; we shall thus obtain

$$\phi = \quad A\epsilon^{mz} \cos (mx - nt)$$

$$\eta = -Amn^{-1} \sin (mx - nt),$$

and the elements of liquid will describe the circles

$$\xi^2 + \zeta^2 = (Am/n)^2 \epsilon^{2mz}.$$

Waves at the Surface of Separation of Two Liquids.

85. Let us first suppose that two liquids of different densities (such as water and mercury) are resting upon one another, and are in repose except for the disturbance produced by the wave motion; also let the liquids be confined between two planes parallel to their surface of separation. Let ρ, ρ' be the densities of the lower and upper liquids respectively, h, h' their depths, and let the origin be taken in the surface of separation when in repose.

In the lower liquid let

$$\phi = A \cosh m (z + h) \cos (mx - nt),$$

and in the upper let

$$\phi' = A' \cosh m (z - h') \cos (mx - nt)$$

also let $$\eta = a \sin (mx - nt)$$

be the equation of the surface of separation. At this surface, the condition that the two liquids should remain in contact requires that

$$d\eta/dt = d\phi/dz = d\phi'/dz, \text{ when } z = 0.$$

Whence $$-na = mA \sinh mh = -mA' \sinh mh'.$$

If δp, $\delta p'$ be the increments of the pressure due to the wave motion just below and just above the surface of separation, then

$$\delta p + g\rho\eta + \rho d\phi/dt = 0,$$

and $$\delta p' + g\rho'\eta + \rho' d\phi'/dt = 0,$$

and since $\delta p = \delta p'$, we obtain

$$\begin{aligned} g(\rho - \rho')\eta &= -\rho d\phi/dt + \rho' d\phi'/dt \\ &= n(-A\rho \cosh mh + A'\rho' \cosh mh') \sin (mx - nt) \\ &= (\rho \coth mh + \rho' \coth mh')\, n^2\eta/m, \end{aligned}$$

whence

$$U^2 = (n/m)^2 = \frac{g(\rho - \rho')}{m(\rho \coth mh + \rho' \coth mh')} \quad \text{.................. (18),}$$

where $m = 2\pi/\lambda$.

86. When λ is small compared with h and h', then mh, mh' are large, and coth mh and coth mh' may be replaced by unity: we thus obtain

$$U^2 = g(\rho - \rho')/m(\rho + \rho').$$

If $\rho' > \rho$, U^2 is negative and therefore n is imaginary; hence if the upper liquid is denser than the lower, the motion cannot be represented by a periodic term in t, and is therefore unstable.

If the density of the upper liquid is small compared with that of the lower, we have approximately

$$U^2 = gm^{-1}(1 - 2\rho'/\rho).$$

If the liquid is water in contact with air, $\rho'/\rho = \cdot 00122$, hence if the air is treated as an incompressible fluid

$$U^2 = \cdot 99756 \times gm^{-1}.$$

87. Secondly, let us suppose that the upper liquid is moving with velocity V', and the lower with velocity V; then we may put

$$\phi = Vx + A \cosh m(z+h) \cos(mx - nt)$$
$$\phi' = V'x + A' \cosh m(z-h') \cos(mx - nt).$$

Let the equation of the surface of separation be

$$F = \eta - a \sin(mx - nt) = 0.$$

Then in both liquids F must be a bounding surface, and therefore by § 12 equation (17), when $z = 0$,

$$\frac{dF}{dt} + \frac{d\phi}{dx}\frac{dF}{dx} + \frac{d\phi}{dz}\frac{dF}{d\eta} = 0,$$
$$\frac{dF}{dt} + \frac{d\phi'}{dx}\frac{dF}{dx} + \frac{d\phi'}{dz}\frac{dF}{d\eta} = 0.$$

Whence $\quad an - mVa + mA \sinh mh = 0,$

$$an - mV'a - mA' \sinh mh' = 0.$$

Hence if $U = n/m$ be the velocity of propagation,

$$A \sinh mh = a(V - U)$$
$$A' \sinh mh' = -a(V' - U).$$

If δp, $\delta p'$ be the increments of pressure at the surface of separation due to the wave motion

$$\delta p/\rho + g\eta + d\phi/dt + \tfrac{1}{2}\{V - Am \cosh mh \cos(mx - nt)\}^2 = \tfrac{1}{2}V^2,$$
$$\delta p'/\rho' + g\eta + d\phi'/dt + \tfrac{1}{2}\{V' - A'm \cosh mh' \cos(mx - nt)\}^2 = \tfrac{1}{2}V'^2.$$

Therefore since $\delta p = \delta p'$,

$$ag(\rho - \rho') = Am\rho(V - U)\cosh mh - A'm\rho'(V' - U)\cosh mh'$$

or $g(\rho - \rho') = m\rho(V - U)^2 \coth mh + m\rho'(V' - U)^2 \coth mh' \ldots (19)$, which determines U.

Stability and Instability.

88. We shall now consider a question which has excited a good deal of attention of late years, viz. the stability or instability of fluid motion.

If a disturbance be communicated to the two liquids which are considered in §§ 85—87, the surface of separation may be conceived to be initially of the form $\eta = a \sin mx$ or $a \cos mx$, where m is a given real quantity, whose value depends upon the nature of the disturbance. An equation of this kind does not of

course represent the most general possible kind of disturbance, but inasmuch as by Fourier's theorem, any arbitrary function can be expressed in the form of a series of sines or cosines, or by a definite integral involving such quantities, an equation of this form is sufficient for our purpose.

We have pointed out that the object of the wave motion problem is to determine n; if therefore n should be found to be a real quantity, the subsequent motion will be periodic, and therefore stable; but if n should turn out to be an imaginary or a complex quantity, the final solution will involve real exponential quantities, and therefore the motion will tend to increase with the time and will be unstable.

To understand this more clearly, it must be recollected that we have neglected the quadratic terms in the equations of motion. The validity of this hypothesis depends firstly on the condition that the disturbance is a small quantity, from which it follows that the *initial* displacements and velocities must also be small quantities; secondly, on the condition that these quantities remain small during the subsequent motion. If the solution thus obtained consists of periodic terms whose amplitudes are small quantities of the same order as the disturbance, the second condition is fulfilled, and the system oscillates about its undisturbed configuration; but if the solution should contain an exponential term of the form ϵ^{kt}, where k is positive, the system will tend to depart from its undisturbed configuration, and the solution will only represent the state of things *in the beginning of the disturbed motion*; and the subsequent history of the motion cannot be ascertained without finding the *complete* solution of the equations of motion, in which the quadratic terms are taken into account. In the former case the motion is stable, and in the latter unstable.

To express this analytically, let us employ complex quantities, and assume that the initial form of the free surface is the real part of

$$\eta = (A - \iota B)\,\epsilon^{\iota mx},$$

where A, B and m are real; and let n be of the form $\alpha + \iota\beta$. Since the form of the free surface at any subsequent time is

$$\eta = (A - \iota B)\,\epsilon^{\iota(mx - nt)},$$

this becomes
$$\eta = (A - \iota B)\,\epsilon^{\iota(mx - \alpha t) + \beta t},$$

the real part of which is

$$\eta = \epsilon^{\beta t}\{A\cos(mx - \alpha t) + B\sin(mx - \alpha t)\} \quad\ldots\ldots (20).$$

If therefore β is positive, the amplitude will tend to increase with the time, and the motion will be unstable. In such a case the two liquids will, after a short time, become mixed together, and will usually remain permanently mixed, if they are capable of mixing; but if they are incapable of remaining permanently mixed, the lighter liquid will gradually work its way upwards, and a stable condition will ultimately be arrived at.

89. If one liquid is resting upon another, equilibrium is possible when the heavier liquid is at the top, but in this case the equilibrium is unstable; for since $\rho' > \rho$, it follows from (18) that n^2 is negative and therefore n is of the form $\pm \iota\beta$. Hence in the beginning of the disturbed motion, the free surface is of the form

$$\eta = A\epsilon^{\pm\beta t}\cos(mx - e).$$

If the upper liquid is moving with velocity V', and the lower with velocity V, the values of U or n/m are determined by the quadratic (19); and the condition of stability requires that the two roots of this quadratic should be real.

Putting k, k' for $m\coth mh$ and $m\coth mh'$, (19) becomes

$$k\rho(V - U)^2 + k'\rho'(V' - U)^2 = g(\rho - \rho').$$

The condition that the roots of this quadratic in U should be real, is

$$g(k\rho + k'\rho')(\rho - \rho') - kk'\rho\rho'(V - V')^2 > 0.$$

It therefore follows that if $\rho > \rho'$, that is if the lower liquid is denser than the upper liquid, the motion *may be stable*. But if $\rho' > \rho$; or if no forces are in action, so that $g = 0$, the motion will be unstable.

90. If no forces are in action, and both liquids are of unlimited extent so that $h = h' = \infty$, the equation for determining U becomes

$$\rho(V - U)^2 + \rho'(V' - U)^2 = 0,$$

the roots of which are

$$U = \frac{\rho V + \rho' V' \pm \iota\sqrt{\rho\rho'}(V - V')}{\rho + \rho'} \quad\ldots\ldots\ldots\ldots (21).$$

Hence U, and therefore n, is a complex quantity, and we may therefore put

$$U = \alpha \pm \iota\beta = n/m,$$

where α and β are determined from (21). If therefore the initial form of the free surface is

$$\eta = a\epsilon^{\iota mx},$$

its form at any subsequent time may be written

$$\eta = \epsilon^{\iota m(x-\alpha t)} \{a'\epsilon^{m\beta t} + b'\epsilon^{-m\beta t}\} \qquad (22),$$

where $a' + b' = a$.

If there is no initial displacement, $\dot{\eta} = 0$ when $t = 0$, in which case $a' = b' = \frac{1}{2}a$. To express this result in real quantities, let $a = A - \iota B$, and (22) becomes

$$\eta = \{A \cos m(x - \alpha t) + B \sin m(x - \alpha t)\} \cosh m\beta t,$$

corresponding to an initial displacement

$$\eta = A \cos mx + B \sin mx.$$

91. When the initial velocity is zero there are three cases worthy of notice.

(i) Let $\rho = \rho'$, $V = -V'$, so that the densities of the two liquids are equal, and their undisturbed velocities are equal and opposite; then from (21), $\alpha = 0$, $\beta = V$, whence

$$\eta = (A \cos mx + B \sin mx) \cosh mVt.$$

(ii) Let $\rho = \rho'$, $V' = 0$, then $\alpha = \frac{1}{2}V$, $\beta = \frac{1}{2}V$, whence

$$\eta = \{A \cos m(x - \tfrac{1}{2}Vt) + B \sin m(x - \tfrac{1}{2}Vt)\} \cosh \tfrac{1}{2}mVt,$$

hence the waves travel in the direction of the stream and with half its velocity.

(iii) Let $\rho = \rho'$, $V = V'$. In this case the roots are equal, but the general solution may be obtained from (21) by putting $V' = V(1 + \gamma)$, where γ ultimately vanishes. We thus obtain

$$\alpha = V + \tfrac{1}{2}V\gamma, \quad \beta = -\tfrac{1}{2}V\gamma,$$

and therefore since γ is small, (22) may be written

$$\eta = \epsilon^{\iota m(x-Vt)} \{a + \tfrac{1}{2}mV\gamma t[a(1-\iota) - 2a']\}.$$

Putting $c = \frac{1}{2}mV\gamma\{a(1-\iota) - 2a'\}$, this becomes

$$\eta = (a + ct)\,\epsilon^{\iota m(x-Vt)}.$$

Let $a = A - \iota B$, $c = C - \iota D$, then the real part is

$$\eta = (A + Ct)\cos m(x - Vt) + (B + Dt)\sin m(x - Vt),$$

corresponding to an initial displacement $\eta = A\cos mx + B\sin mx$. If the initial velocity $\dot{\eta}$ is zero, $C = mBV$, $D = -mAV$, and

$$\eta = (A + mBVt)\cos m(x - Vt) + (B - mAVt)\sin m(x - Vt).$$

The peculiarity of this solution is, that previously to displacement there is no real surface of separation at all. Hence if we have a thin surface dividing the air, such as a flag whose inertia may be neglected, it appears from the last equation that (neglecting changes in the density of the air), the motion of the flag will be unstable and that it will flap.

Long Waves in Shallow Water.

92. In the theory of long waves it is assumed, that the lengths of the waves are so great in proportion to the depth of the water, that the vertical component of the velocity can be neglected, and that the horizontal component is uniform across each section of the canal. In § 82 we saw that if the depth is small compared with the wave-length, then $U^2 = gh$, provided the square of the velocity is neglected. We shall now examine this result in connection with the above-mentioned assumption.

Let the motion be made steady by impressing on the whole liquid a velocity equal and opposite to the velocity of propagation of the waves. Let η be the elevation of the liquid above the undisturbed surface; U, u the velocities corresponding to h and $h + \eta$ respectively. The equation of continuity gives

$$u = hU/(h + \eta),$$

whence
$$U^2 - u^2 = U^2(2h\eta + \eta^2)/(h + \eta)^2.$$

If δp be the excess of pressure due to the wave motion

$$\delta p = \left\{\frac{U^2(2h + \eta)}{2(h + \eta)^2} - g\right\}\rho\eta.$$

When η/h is very small, the quantity in brackets is $U^2/h - g$; whence if $U^2 = gh$, the change of pressure at a height $h + \eta$ vanishes to a first approximation, and therefore a free surface is possible.

If the condition $U^2 = gh$ is satisfied, the change of pressure to a second approximation is

$$\delta p = -3g\rho\eta^2/2h,$$

which shows that the pressure is defective at all parts of the wave at which η differs from zero. *Unless therefore* η^2 *can be neglected, it is impossible to satisfy the condition of a free surface for a stationary long wave;—in other words, it is impossible for a long wave of finite height to be propagated in still water without change of type.* If however η be everywhere positive, a better result can be obtained with a somewhat increased value of U; and if η be everywhere negative, with a diminished value. We therefore infer that waves of elevation travel with a somewhat higher, and waves of depression with a somewhat lower, velocity than that due to half the undisturbed depth[1].

93. The theory of long waves in a canal may be investigated analytically as follows[2].

Let the origin be in the bottom of the liquid, h the undisturbed depth, η the elevation; and let x be the abscissa of an element of liquid when undisturbed, ξ the horizontal displacement. The quantity of liquid originally between the planes x and $x+dx$ is hdx; at the end of an interval t, the breadth of this stratum is $dx\,(1 + d\xi/dx)$, and its height is $h+\eta$, whence the equation of continuity is

$$(1 + d\xi/dx)\,(h+\eta) = h \quad\text{.................}\quad (23).$$

Let us now investigate the motion of a column of liquid contained between the planes whose original distance was dx; and let us suppose that in addition to gravity, small horizontal and vertical disturbing forces X and Y act. Since the vertical acceleration is neglected, the pressure will be equal to the hydrostatic pressure due to a column of liquid of height $h+\eta$, whence

$$p = g\rho\,(h+\eta-y) + \rho\int_y^{h+\eta} Y\,dy \quad\text{...........}\quad (24).$$

The equation of motion of the stratum is

$$\rho h\frac{d^2\xi}{dt^2} = -\frac{dp}{dx}(h+\eta) + X\rho h \quad\text{..............}\quad (25).$$

Now from (24),

$$\frac{dp}{dx} = g\rho\frac{d\eta}{dx} + \rho Y\frac{d\eta}{dx} + \rho\int_y^{h+\eta}\frac{dY}{dx}\,dy \quad\text{........}\quad (26);$$

[1] Lord Rayleigh, "On Waves," *Phil. Mag.* April, 1876.

[2] Airy, "Tides and Waves," *Encyc. Met.*

also in most problems to which the theory applies, the last two terms on the right-hand side of (26) are very much smaller than the first, and may therefore be neglected, whence (25) becomes

$$h\frac{d^2\xi}{dt^2} = -g(h+\eta)\frac{d\eta}{dx} + Xh.$$

Substituting the value of η from (23) we obtain

$$\frac{d^2\xi}{dt^2} = gh\frac{d^2\xi}{dx^2}\left(1+\frac{d\xi}{dx}\right)^{-3} + X \text{ .... } (27).$$

For a first approximation, we may neglect squares and products of small quantities, and (23) and (27) respectively become

$$\eta/h = -d\xi/dx \text{ .... } (28),$$

$$\frac{d^2\xi}{dt^2} = gh\frac{d^2\xi}{dx^2} + X \text{ .... } (29).$$

In order to solve (29) when $X=0$, assume $\xi = \epsilon^{\iota(mx-nt)}$, and we obtain $n/m = (gh)^{\frac{1}{2}}$, which shows that the velocity of propagation is equal to $(gh)^{\frac{1}{2}}$.

Stationary Waves in Flowing Water[1].

94. Let us suppose that water is flowing uniformly along a straight canal with vertical sides, and that between two points A and B there are small inequalities, and that beyond these points the bottom is perfectly level. Let a be the depth, u the velocity, p the *mean* pressure beyond A; b the depth, v the velocity, and q the *mean* pressure beyond B: also let f be the difference of levels of the bottom at A and B.

The total energy of the liquid per unit of the canal's length and breadth, at points beyond B, is

$$\tfrac{1}{2}v^2b + g\int_0^b y\,dy + w = \tfrac{1}{2}(v^2+gb)\,b + w,$$

where w is the wave energy, and the density of the liquid is taken as unity. At very great distances beyond B the wave motion will have subsided and w will be zero.

The equation of continuity is

$$au = bv = M \text{ .... } (30).$$

[1] Sir W. Thomson, *Phil. Mag.* (5) vol. XXII. 353.

The dynamical equation is found from the consideration, that the difference between the work done by the pressure p upon the volume of water entering at A, and the work done by the pressure q at B upon an equal volume of water passing away at B, is equal to the difference between the energy which passes away at B, and the energy which enters at A. Whence

$$pau - qbv = (\tfrac{1}{2}v^2b + \tfrac{1}{2}gb^2 + w)\,v - (\tfrac{1}{2}u^2a + g\int_f^{a+f} ydy)\,u,$$

which by (30) becomes,

$$p - q = \tfrac{1}{2}v^2 + \tfrac{1}{2}gb + w/b - \tfrac{1}{2}u^2 - g\,(f + \tfrac{1}{2}a) \ldots\ldots (31).$$

Now p and q are the *mean* pressures, and therefore since the pressure at the free surface is zero,

$$p = \tfrac{1}{2}ga, \quad q = \tfrac{1}{2}gb + w'/b,$$

where w' denotes a quantity depending on the wave disturbance; whence (31) becomes

$$\tfrac{1}{2}M^2(a^2 - b^2)/a^2b^2 - g\,(a - b + f) + (w - w')/b = 0 \ldots\ldots (32).$$

If we put

$$D^3 = 2a^2b^2/(a + b), \quad M = VD;$$

D will denote a mean depth intermediate between a and b, and approximately equal to their arithmetic mean when their difference is small in comparison with either; and V will similarly denote a corresponding mean velocity of flow. We thus obtain from (32)

$$b - a = \frac{f - (w - w')/gb}{1 - V^2/gD}.$$

If $b - a$ were exactly equal to f, and there were no disturbance of the water beyond B, the mean level of the water would be the same at great distances beyond A and B; but if this is not the case, there will be a rise or fall of level, determined by the formula

$$y = b - a - f = \frac{V^2f/gD + (w - w')/gb}{1 - V^2/gD}.$$

Let us now suppose that between A and B there are various small inequalities; each of these inequalities will produce small waves whose nature is determined by the form of the functions w, w'; hence w and w' will both be small quantities and the sign of y will be independent of that of $w - w'$. Now f is positive or negative according as the bottom at A is higher or lower than the

bottom at B. *Hence if $V^2 < gD$ the upper surface of the water rises when the bottom falls, and falls when the bottom rises; and the converse is the case when $V^2 > gD$.*

Theory of Group Velocity.

95. When a group of waves advances into still water, it is observed that the velocity of the group is less than that of the individual waves of which it is composed. This phenomenon was first explained by Sir G. Stokes[1], who regarded the group as formed by the superposition of two infinite trains of waves of equal amplitudes and nearly equal wave-lengths, advancing in the same direction.

Let the two trains of waves be represented by $\cos k(Vt - x)$ and $\cos k'(V't - x)$; their resultant is equal to

$$\cos k(Vt - x) + \cos k'(V't - x) = 2\cos \tfrac{1}{2}\{(k'V' - kV)t - (k' - k)x\}$$
$$\times \cos \tfrac{1}{2}\{(k'V' + kV)t - (k' + k)x\}.$$

If $k' - k$, $V' - V$ be small, this represents a train of waves whose amplitude varies slowly from one point to another between the limits 0 and 2, forming a series of groups separated from one another by regions comparatively free from disturbance. The position at time t of the middle of the group, which was initially at the origin, is given by

$$(k'V' - kV)t - (k' - k)x = 0,$$

which shows that the velocity of propagation U of the group is

$$U = (k'V' - kV)/(k' - k).$$

In the limit when the number of waves in each group is indefinitely great we have $k' = k + \delta k$, $V' = V + \delta V$, whence

$$U = \frac{d(kV)}{dk}.$$

Capillary Waves.

96. Most liquids which are incapable of remaining permanently mixed, exhibit a certain phenomenon called capillarity[2],

[1] *Smith's Prize Examination*, 1876; and Lord Rayleigh, "On Progressive Waves"; *Proc. Lond. Soc.* vol. IX.

[2] The reader who desires to study the theory of Capillarity is recommended

when in contact with one another. This phenomenon can be explained by supposing that the surface of separation is capable of sustaining a tension, which is equal in all directions, and is independent of the form of the surface of separation.

The surface tension depends upon the nature of both the liquids which are in contact with one another. Thus at a temperature of 20° C., the surface tension of water in contact with air is 81 dynes per centimetre; whilst the surface tension of water in contact with mercury is 418 dynes per centimetre.

The surface tension diminishes as the temperature increases; also a surface tension cannot exist at the common surface of two liquids, such as water and alcohol, which are capable of becoming permanently mixed.

97. We shall now consider the effect of surface tension upon the propagation of waves.

Let T be the surface tension, and let p and $p+\delta p$ be the pressures just outside and just inside the free surface of a liquid; then

$$\delta p/\rho + g\eta + \dot{\phi} = 0 \quad \ldots\ldots \ldots\ldots\ldots\ldots (33).$$

But if we resolve the forces which act upon a small element δs of the free surface vertically, and neglect the vertical acceleration, and put $\delta\chi$ for the angle which δs subtends at the centre of curvature, we obtain

$$\delta p \delta s = T\delta\chi,$$

whence

$$\delta p = T\frac{d\chi}{ds}.$$

Now

$$\frac{d\eta}{dx} = \cot\chi,$$

therefore

$$\frac{d^2\eta}{dx^2} = -\operatorname{cosec}^2\chi\frac{d\chi}{ds}.$$

Since χ is nearly equal to $\frac{1}{2}\pi$, we may put $\operatorname{cosec}\chi = 1$, and $ds = dx$, whence

$$\delta p = -T\frac{d^2\eta}{dx^2}.$$

to consult Chapter xx. of Maxwell's *Heat*; and also the article on Capillarity in the *Encyclopædia Britannica* by the same author.

A table of the superficial tensions of various liquids will be found in Everett's *Units and Physical Constants*, p. 49.

Substituting in (33), differentiating the result with respect to t, and remembering that $\dot{\eta} = d\phi/dz$, and that $d^2\phi/dx^2 = -d^2\phi/dz^2$, we obtain

$$\frac{T}{\rho}\frac{d^3\phi}{dz^3} + g\frac{d\phi}{dz} + \frac{d^2\phi}{dt^2} = 0 \text{ ....(34).}$$

This is the condition to be satisfied at the free surface.

98. We shall now apply the preceding result to determine the capillary waves propagated in an ocean of depth h.

Let $\qquad \phi = A \cosh m(z+h) \cos(mx - nt).$

Substituting in (34) we obtain

$$Tm^3/\rho + mg = n^2 \coth mh,$$

whence

$$U^2 = n^2/m^2 = (g\lambda/2\pi + 2\pi T/\rho\lambda) \tanh 2\pi h/\lambda \text{......(35).}$$

Equation (35) determines the velocity of propagation corresponding to a given wave-length.

99. Let us now suppose that the depth h is so great that mh may be treated as infinite; then $\coth mh = \pm 1$ according as m is positive or negative. Hence it will be sufficient for our purpose to discuss the equation

$$Tm^3/\rho + mg = n^2 \text{......................(36),}$$

when m is positive.

When $m = 0$, $n = 0$; and as m increases from zero to infinity, the value of n^2 is always positive, and consequently periodic motion is possible for all values of m, that is for all wave-lengths.

If a given value be assigned to n, (36) is a cubic for determining m. The real root is obviously positive; and since the discriminant[1] of the cubic is positive, the other two roots are complex. Hence there is only one wave-length corresponding to any given period.

[1] It can be shown by means of Taylor's theorem, that, if $f(x)$ be any rational algebraic function of x, the condition that the equation $f(x) = 0$ should have a pair of equal roots is obtained by eliminating x between $f(x) = 0$ and $f'(x) = 0$. The result of the elimination is a certain function of the coefficients which is called the *discriminant* of $f(x)$; and it is shown in treatises on Algebra that two of the roots of a cubic will be real, equal or complex, according as the discriminant is negative, zero or positive. If the cubic be $Ax^3 + Bx^2 + Cx + D$, the discriminant is

$$27A^2D^2 + 4B^3D + 4AC^3 - 18ABCD - B^2C^2.$$

If U be the velocity of propagation, $n = mU$; whence (36) becomes

$$Tm^2/\rho - mU^2 + g = 0 \qquad (37).$$

Hence $U = \infty$, when $m = 0$ and $m = \infty$; and therefore U must be a minimum for some intermediate value of m, which by ordinary methods can be shown to be given by the equation $U^2 = 2Tm/\rho$. By means of this result, the minimum velocity of propagation and the corresponding wave-length can be shown to be given by the equations

$$U = (4Tg/\rho)^{\frac{1}{4}}, \quad \lambda = 2\pi (T/g\rho)^{\frac{1}{2}} \qquad (38).$$

A wave whose length is less than the preceding critical value of λ is called by Lord Kelvin a *ripple*[1]. Now if we write (36) in the form

$$U^2 = g\lambda/2\pi + 2\pi T/\rho\lambda \qquad (39),$$

it follows that when λ is given by (38) both terms on the right-hand side are equal; also for long waves the first term is the most important, whilst for short waves the second is the most important. Hence the effect of gravity is most potent in producing long waves, and the effect of surface tension in producing ripples.

100. In § 85 we have considered the propagation of waves at the surface of separation of two liquids, which are moving with different velocities. We shall now consider the production of ripples by wind blowing over the surface of still water.

Let V be the velocity of the wind, which is supposed to be parallel to the undisturbed surface of the water, σ the density of air referred to water.

Since the changes of density of the air are very small in the neighbourhood of the water, the air may approximately be regarded as an incompressible fluid, whence if the accented letters refer to the water, the kinematical conditions at the boundary give

$$\phi = Vx + a(U - V)\epsilon^{-mz}\cos(mx - nt),$$
$$\phi' = -aU\epsilon^{mz}\cos(mx - nt),$$

where U is the velocity of propagation of the waves in the water, and $\eta = a\sin(mx - nt)$ is the equation of its free surface.

Since the vertical acceleration is neglected, the dynamical condition at the free surface is

$$T\delta\chi + (\delta p - \delta p')\,\delta s = 0,$$

[1] *Phil. Mag.* (4) vol. XLII.

or $$\delta p - \delta p' = T\frac{d^2\eta}{dx^2} \text{ .... (40).}$$

Now

$$\delta p + g\sigma\eta + \dot{\phi} + \tfrac{1}{2}\{V - am(U-V)\sin(mx-nt)\}^2 - \tfrac{1}{2}V^2 = 0,$$

or $$\delta p + a\sigma\{g + n(U-V) - m(U-V)\}\sin(mx-nt) = 0.$$

Similarly

$$\delta p' + (g - Un)\,a\sin(mx-nt) = 0,$$

whence (40) becomes

$$Tm^2 - mU^2 - \sigma m(U-V)^2 + (1-\sigma)g = 0 \text{ ....(41).}$$

Putting $U = n/m$ this may be written

$$Tm^3 - \sigma V^2m^2 + 2\sigma Vnm + g(1-\sigma)m = (1+\sigma)n^2 \text{...(42).}$$

Let W be the velocity of propagation of waves in water when there is no wind; then writing $V = 0$, $U = W$ in (41) we obtain

$$Tm^2 - (1+\sigma)W^2m + (1-\sigma)g = 0 \text{ ....(43).}$$

The minimum value of W is given by the equation

$$W^2 = \frac{2}{1+\sigma}\sqrt{Tg(1-\sigma)} \text{ ....(44),}$$

and the corresponding value of λ is

$$\lambda = 2\pi\sqrt{T/g(1-\sigma)} \text{ ....(45).}$$

Multiply (43) by m and subtract from (42) and we shall obtain,

$$\left\{W^2 - \frac{\sigma V^2}{(1+\sigma)^2}\right\}m^2 = \left(n - \frac{\sigma Vm}{1+\sigma}\right)^2 \text{ ....(46).}$$

Hence if

$$W^2 < \frac{\sigma V^2}{(1+\sigma)^2}$$

the value of n will be complex, and periodic motion will be impossible. Equation (44) gives the minimum value of W; hence in order that wave-motion may be possible for waves of all lengths, we must have

$$V^2 < \frac{2(1+\sigma)}{\sigma}\sqrt{Tg(1-\sigma)} \text{ ....(47).}$$

When (47) is satisfied, (46) may be written

$$U = \frac{\sigma V}{1+\sigma} \pm \sqrt{\left\{W^2 - \frac{\sigma V^2}{(1+\sigma)^2}\right\}} \text{ .... (48).}$$

We shall now discuss this equation.

Case (i). $V < W\sqrt{(1+\sigma)/\sigma}$.

In this case one of the values of U is positive and the other negative; hence waves can travel either with or against the wind. Moreover since the positive value is numerically greater than the negative value, waves travel faster with the wind than against the wind; also the velocity of waves travelling against the wind is always less than W.

Case (ii). $V > W\sqrt{(1+\sigma)/\sigma}$.

In this case both values of U are positive; hence waves cannot travel against the wind.

Case (iii). When $V < 2W$, the velocity of waves travelling with the wind is $> W$; when $V > 2W$ this velocity is $< W$; and when $V = 2W$, the velocity of waves travelling with the wind is undisturbed.

EXAMPLES.

1. A liquid of infinite depth is bounded by a fixed plane perpendicular to the direction of propagation of the waves. Prove that each element of liquid will vibrate in a straight line, and draw a figure representing the free surface and the direction of motion of the elements, when the crest of the wave reaches the fixed plane.

2. Prove that the velocity of propagation of long waves in a semicircular canal of radius a and whose banks are vertical, is

$$\tfrac{1}{2}(\pi g a)^{\frac{1}{2}}.$$

3. If two series of waves of equal amplitude and nearly equal wave-length travel in the same direction, so as to form alternate lulls and roughness, prove that in deep water these are propagated with half the velocity of the waves; and that as the ratio of the depth to the wave-length decreases from ∞ to 0, the ratio of the two velocities of propagation increases from $\frac{1}{2}$ to 1.

4. If a small system of rectilinear waves move parallel to and over another large rectilinear system, prove that the path of a particle of water is an epicycloid or hypocycloid, according as the two systems are moving in the same or opposite directions.

5. A fine tube made of a thin slightly elastic substance is filled with liquid; prove that the velocity of propagation of a disturbance in the liquid is $(\lambda\theta/a\rho)^{\frac{1}{2}}$, where a is the internal diameter of the tube, θ its thickness, λ the coefficient of elasticity of the material of which it is made, and ρ the density of the liquid.

6. A horizontal rectangular box is completely filled with three liquids which do not mix, whose densities reckoned downwards are σ_1, σ_2, σ_3, and whose depths when in equilibrium are l_1, l_2, l_3 respectively. Show that if long waves are propagated at their common surfaces, the velocity of propagation V must satisfy the equation

$$\{(\sigma_1/l_1+\sigma_2 l_2)V^2-g(\sigma_2-\sigma_1)\}\{(\sigma_2/l_2+\sigma_3 l_3)V^2-g(\sigma_3-\sigma_2)\}=\sigma_2^2V^4/l_2^2.$$

7. Prove that liquid of density ρ flowing with mean velocity U through an elastic tube of radius a, will throw the surface into slight stationary corrugations, of which the number per unit of length is

$$(2\rho a U^2-\lambda)^{\frac{1}{2}}/(2\pi a T)^{\frac{1}{2}},$$

where λ is the modulus of elasticity of the substance of the tube, and T its total tension.

8. Prove that the velocity potential

$$\phi = A(\lambda + 2\pi^2y^2/\lambda)\sin 2\pi(vt-x)/\lambda$$

satisfies the equation of continuity in a mass of water, provided the ratio y/λ is so small for all possible values of y, that its square may be neglected. Hence prove that if the water in a canal of uniform breadth and uniform depth k, be acted upon in addition to gravity by the horizontal force $Ha^{-1}\sin 2(mt-x/a)$, where H and m are small and a is large, the equation of the free surface may be of the form

$$y = k + \frac{Hk}{2(gk-m^2a^2)}\cos 2(mt-x/a).$$

9. Two liquids of density ρ, ρ' completely fill a shallow pipe; prove that the velocity of propagation of long waves is

$$U^2=\frac{g(\rho-\rho')AA'}{b(A'\rho+A\rho')},$$

where A, A' are the areas of the vertical sections of the two liquids when undisturbed, and b is the breadth of the surface of separation.

10. If the upper liquid were moving with mean velocity U', and the lower with mean velocity U, and there is a surface tension T, prove that the wave-length is determined by the equation

$$4T\pi^2/\lambda^2 = b(\rho U^2/A + \rho' U'^2/A') - g(\rho - \rho').$$

11. If the bottom of a horizontal canal of depth h be constrained to execute a simple harmonic motion, such that the vertical displacement at a distance x from a given line across the canal and perpendicular to its length, be given by $k \cos m(x - vt)$, k being small; show that when the motion is steady, the form of the free surface is given by

$$y = h + \frac{kv^2}{v^2 - gh} \cos m(x - vt).$$

12. A shallow trough is filled with oil and water, the depth of the water being k and its density σ, and that of the oil being h and its density ρ. Prove that the velocity of propagation v of long waves is

$$v^2/g = \tfrac{1}{2}(h + k) + \tfrac{1}{2}\{(h - k)^2 + 4hk\rho/\sigma\}^{\frac{1}{2}}.$$

(Note that there may be slipping between the oil and water.)

13. If water is flowing with velocity proportional to the distance from the bottom, V being the velocity of the stream at its surface, prove that the velocity of propagation U of waves in the direction of the stream is given by

$$(U - V)^2 - V(U - V)W^2/gh - W = 0,$$

where W is the velocity of propagation of waves in still water.

14. Two liquids of densities ρ, ρ', each of which half fills a pipe of which the cross section is a square with a vertical diagonal of length $2h$, are slightly disturbed. Neglecting the disturbing effect of the boundary in the neighbourhood of the surface of separation, prove that the velocity of propagation of progressive waves along the pipe is given by the equation

$$U^2 = \frac{g(\rho - \rho')}{2m(\rho + \rho')}(\tanh \textit{ or } \coth)\, mh.$$

CHAPTER V.

RECTILINEAR VORTEX MOTION.

101. THE present Chapter will be devoted to the consideration of certain problems of two-dimensional motion, which involve molecular rotation.

A vortex line may be defined to be a line whose direction coincides with the direction of the instantaneous axis of molecular rotation. Hence the differential equations of a vortex line are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}.$$

When the motion is in two dimensions,

$$w = 0, \quad du/dz = 0, \quad dv/dz = 0, \quad \xi = 0, \quad \eta = 0,$$

and therefore the vortex lines are all parallel to the axis of z.

In the case of a liquid, it follows from § 20, equations (27), that the rotation ζ may be any function of x, y and t, which satisfies the equation,

$$\frac{d\zeta}{dt} + u\frac{d\zeta}{dx} + v\frac{d\zeta}{dy} = 0 \quad\quad (1),$$

also a current function always exists, such that

$$u = d\psi/dy, \quad v = -d\psi/dx \quad\quad (2);$$

whilst u, v and ζ are connected together by the equation

$$\frac{dv}{dx} - \frac{du}{dy} = 2\zeta \quad\quad (3).$$

Substituting for u, v in terms of ψ, we obtain

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + 2\zeta = 0 \quad\quad (4).$$

Substituting from (2) and (4) in (1), we obtain

$$\left(\frac{d}{dt}+\frac{d\psi}{dy}\frac{d}{dx}-\frac{d\psi}{dx}\frac{d}{dy}\right)\nabla^2\psi=0 \quad\ldots\ldots\ldots\ldots\ldots(5).$$

From this equation it follows that the molecular rotation is not a quantity which can be chosen arbitrarily; for ψ must satisfy (5), and the corresponding value of ζ is then determined by (4).

When the motion is steady, none of the quantities are functions of t, and we obtain from (1) by Lagrange's method

$$2\zeta=F'(\psi),$$

where F is an arbitrary function, which agrees with § 24. The pressure is determined by (31) of the same section.

The current function at all points of the rotationally moving liquid is now determined by the equation

$$\nabla^2\psi+F'(\psi)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

At every point of the irrotationally moving liquid which surrounds the vortices, $\zeta=0$, and therefore

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Equations (4) and (7) show, that ψ is the potential of indefinitely long cylinders composed of attracting matter of density $\zeta/2\pi$, which occupy the same positions as the vortices.

102. The integral $\iint\zeta dxdy$ is called the *vorticity* of the mass of rotationally moving liquid; and we shall now show that the vorticity is an absolute constant.

Draw any closed curve which completely surrounds all the rotationally moving liquid and does not cut any of it. Then since ζ is zero at all points of the liquid where the motion is irrotational, it follows that if τ denote the vorticity,

$$\tau=\iint\zeta dxdy,$$

where the integration extends over the whole area enclosed by the curve. Substituting the value of ζ from (3) we obtain by Green's theorem,

$$2\tau=\iint\left(\frac{dv}{dx}-\frac{du}{dy}\right)dxdy$$

$$=\int(udx+vdy),$$

where the line integral extends round the closed curve. By § 27, this line integral is equal to the *circulation* κ due to the whole of the rotationally moving liquid within the curve; and by the same article, the circulation has been shown to be constant. Hence the vorticity is constant and equal to half the circulation.

103. We shall now consider the steady motion in an infinite liquid of a single rectilinear vortex, whose cross section is a circle of radius a, and whose molecular rotation is constant.

In order that the cross section may remain circular, it is necessary that ψ should be a function of r alone.

Denoting the values of quantities inside the vortex by accented letters, equations (4) and (7) become

$$\frac{d^2\psi'}{dr^2} + \frac{1}{r}\frac{d\psi'}{dr} + 2\zeta = 0 \quad\text{.....................(8)},$$

which gives the value of ψ inside the vortex, and

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = 0 \quad\text{........................(9)},$$

which gives the value outside.

The complete integrals of (8) and (9) are

$$\psi' = A\log r + B - \tfrac{1}{2}\zeta r^2$$

and
$$\psi = C\log r + D.$$

Now ψ' must not be infinite when $r = 0$, and therefore $A = 0$; also at the boundary of the vortex, where $r = a$,

$$\psi' = \psi, \qquad d\psi'/dr = d\psi/dr;$$

whence
$$B - \tfrac{1}{2}\zeta a^2 = C\log a + D$$
$$-\zeta a^2 = C,$$

and therefore
$$C = -\zeta a^2 = -\zeta\sigma/\pi = -m/\pi,$$

where σ is the area of the cross section, and m is the vorticity of the vortex. The constant D contributes nothing to the velocity, and may therefore be omitted, whence

$$\psi' = \tfrac{1}{2}\zeta(a^2 - r^2) - (m/\pi)\log a \quad\text{..............(10)},$$

$$\psi = -(m/\pi)\log r \quad\text{..........................(11)}.$$

Now $-d\psi/dr$ is the velocity perpendicular to r, whence inside the vortex

$$-d\psi'/dr = \zeta r \quad\text{.......................} \quad (12),$$

which vanishes when $r = 0$, and outside

$$-d\psi/dr = m/\pi r \quad\text{.......................} \quad (13).$$

Hence a single vortex whose cross section is circular, if existing in an infinite liquid, will remain at rest and will rotate as a rigid body. It will also produce at every point of the irrotationally moving liquid with which it is surrounded, a velocity which is perpendicular to the line joining that point with the centre of its cross section, and which is inversely proportional to the distance of that point from the centre.

104. Outside the vortex, where the motion is irrotational, a velocity potential of course exists. To find its value we have

$$\frac{d\phi}{dx} = \frac{d\psi}{dy} = -\frac{my}{\pi r^2}, \qquad \frac{d\phi}{dy} = -\frac{d\psi}{dx} = \frac{mx}{\pi r^2},$$

whence $$\phi = -\frac{m}{\pi}\int \frac{ydx - xdy}{x^2 + y^2} = (m/\pi)\tan^{-1} y/x \quad\text{.........} (14).$$

It therefore follows that ϕ is a many-valued function, whose cyclic constant is $2m$. The circulation, i.e. the line integral $\int(udx + vdy)$, is zero when taken round any closed curve which does not surround the vortex, and is equal to $2m$ when the curve surrounds the vortex; whence if κ be the circulation, $m = \frac{1}{2}\kappa$, and the values of ϕ and ψ may be written

$$\phi = (\kappa/2\pi)\tan^{-1} y/x, \quad \psi = -(\kappa/2\pi)\log r.$$

105. The investigations of the last two articles are kinematical; we shall now calculate the value of the pressure within and without the vortex.

Let the values of the quantities inside the vortex be distinguished from those outside by accented letters.

Outside the vortex

$$p/\rho = C - \dot{\phi} - \tfrac{1}{2}q^2,$$

and since $\dot{\phi} = 0$, and $q = m/\pi r = \kappa/2\pi r$, we obtain

$$p/\rho = C - \kappa^2/8\pi^2 r^2 \quad\text{....................} (15),$$

whence if Π be the pressure at an infinite distance

$$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{\kappa^2}{8\pi^2 r^2} \qquad (16).$$

The equation of motion inside the vortex is

$$\frac{1}{\rho}\frac{dp'}{dr} = \frac{q'^2}{r} = \frac{\kappa^2 r}{4\pi^2 a^4},$$

whence

$$\frac{p'}{\rho} = \frac{\kappa^2 r^2}{8\pi^2 a^4} + \frac{P}{\rho} \qquad (17),$$

where P is the pressure at the centre of the vortex.

At the surface of the vortex where $r = a$, $p = p'$, whence

$$P/\rho = \Pi/\rho - \kappa^2/4\pi^2 a^2 \qquad (18),$$

and therefore

$$\frac{p'}{\rho} = \frac{\Pi}{\rho} - \frac{\kappa^2}{4\pi^2 a^2}\left(1 - \frac{r^2}{2a^2}\right) \qquad (19).$$

Hence if

$$\Pi < \kappa^2\rho/4\pi^2 a^2,$$

p' will become negative for some value of $r < a$, which shows that a cylindrical hollow will exist in the vortex, which is concentric with its outer boundary.

When there is no hollow, equations (16) and (19) show that the pressure is a minimum at the centre of the vortex, where it is equal to $\Pi - \kappa^2\rho/4\pi^2 a^2$, and that it gradually increases until the surface is reached, at which it is equal to $\Pi - \kappa^2\rho/8\pi^2 a^2$, and that it then continues to increase to infinity, where its value is Π.

It is also possible to have a hollow cylindrical space, round which there is cyclic irrotational motion. Such a space is called a *hollow vortex*. The condition for its existence requires that $p = 0$ when $r = a$, and therefore by (16)

$$\Pi = \kappa^2\rho/8\pi^2 a^2.$$

This equation determines the value of the radius of the hollow, when the pressure at a very great distance is given.

106. Kirchhoff has shown that it is possible for a vortex whose cross section is an invariable ellipse, and whose molecular rotation at every point is constant, to rotate in a state of steady motion in an infinite liquid, provided a certain relation exists between the molecular rotation and the angular velocity of the axes of the cross section.

The current function is evidently equal to the potential of an elliptic cylinder of density $\zeta/2\pi$. Let a and b be the semi-axes of the cross section, then the value of ψ inside the vortex may be taken to be

$$\psi' = D - \zeta(Ax^2 + By^2)/(A+B),$$

where A, B, D are constants, for this value of ψ' satisfies (4).

Let $x = c\cosh\eta\cos\xi$, $y = c\sinh\eta\sin\xi$, where $c = (a^2-b^2)^{\frac{1}{2}}$, and let $\eta=\beta$ at the surface; the value of ψ' becomes

$$\psi' = D - \zeta c^2(A\cosh^2\eta\cos^2\xi + B\sinh^2\eta\sin^2\xi)/(A+B).$$

Also let the value of ψ outside the vortex be

$$\psi = A'\epsilon^{-2\eta}\cos 2\xi + D\eta/\beta.$$

When $\eta=\beta$, we must have

$$\psi - \psi' = \text{const.},\quad d\psi/d\eta = d\psi'/d\eta.$$

Therefore $\quad A'\epsilon^{-2\beta} = -\tfrac{1}{2}\zeta c^2(A\cosh^2\beta - B\sinh^2\beta)/(A+B)$

and $\quad A'\epsilon^{-2\beta} = \tfrac{1}{2}\zeta c^2(A-B)\sinh\beta\cosh\beta/(A+B).$

Whence $\quad A'(a-b)^2 = -\dfrac{\zeta c^2(Aa^2-Bb^2)}{2(A+B)} = \dfrac{\zeta c^2(A-B)ab}{2(A+B)}.$

Therefore $Aa = Bb$ and

$$\psi' = D - \zeta(bx^2+ay^2)/(a+b).$$

Let ω be the angular velocity of the axes; u, v the velocities of the liquid parallel to them, then

$$\dot{x} - y\omega = u = d\psi'/dy = -2a\zeta y/(a+b),$$
$$\dot{y} + x\omega = v = -d\psi'/dx = 2b\zeta x/(a+b).$$

The boundary condition is

$$\dot{x}\frac{dF}{dx} + \dot{y}\frac{dF}{dy} = 0,$$

where $F = (x/a)^2 + (y/b)^2 - 1 = 0$. Whence

$$\left(\omega - \frac{2a\zeta}{a+b}\right)\frac{1}{a^2} + \left(\frac{2b\zeta}{a+b} - \omega\right)\frac{1}{b^2} = 0,$$

therefore $\quad \omega = 2ab\zeta/(a+b)^2.$

We therefore obtain

$$\dot{x} = -a\omega y/b,\quad \dot{y} = b\omega x/a,$$

the integrals of which are

$$x = La\cos(\omega t + \alpha),\quad y = Lb\sin(\omega t+\alpha),$$

where L and α are the constants of integration. Whence the path of every particle relatively to the boundary is a similar ellipse.

107. A complete investigation respecting the stability of a vortex is given in my larger treatise; but it may be stated that when the cross section is circular both cases of steady motion which we have considered, viz. the steady motion of a solid vortex, and the steady motion of a hollow vortex, are stable; and consequently if a small disturbance be communicated to either kind of vortex, the vortex will proceed to oscillate about its mean configuration in steady motion, and will not mix with the surrounding liquid. It is otherwise if the liquid composing the vortex is of different density to the surrounding liquid, for in that case the motion will be unstable; and consequently after a sufficient time has elapsed, the two liquids will become mixed together, and will form what has been called a vortex sponge. The stability of Kirchhoff's elliptic vortex does not appear to have been investigated.

The preceding results are however only applicable when there is a single vortex in an infinite liquid, and it is therefore important to enquire, whether the presence of other vortices or the presence of plane or curved boundaries renders the motion unstable. This question has been dealt with by Prof. J. J. Thomson, and he has shown that when there are two rectilinear vortices in a liquid, the linear dimensions of whose cross sections are small in comparison with the shortest distance between them, their cross sections will always remain approximately circular; and it is inferred from this that a similar result holds good in the case of any number of vortices. We therefore conclude that when a number of vortices of small cross section exist in a liquid, they may be treated as if their cross sections remain circular throughout the subsequent motion, provided none of the vortices approach too closely to one another. It therefore follows, that the effect of any number of vortices upon any external point of the liquid is equal to the sum of the effects due to each; so that if $m_1, m_2 \ldots\ldots$ be the vorticities of the vortices, $r_1, r_2 \ldots\ldots$ their distances from any point P of the liquid, the current function due to the whole motion is

$$\psi = -(m_1/\pi)\log r_1 - (m_2/\pi)\log r_2 - \ldots\ldots\ldots\ldots\ldots$$

Moreover since a rectilinear vortex is incapable of producing any motion of translation upon itself, it follows, that the motion of any particular vortex is the same as would be produced by all the other vortices upon the point occupied by the particular vortex, if the latter did not exist.

108. We shall pass on to consider the motion of a number of vortices of small and approximately circular cross sections.

Putting $m/\pi = M$, it follows that since we neglect deformations of the cross sections, the current function due to each vortex will be $-M \log r$, and the velocity due to it at any point P will be M/r, and will be perpendicular to the line joining P with the vortex. Hence if two vortices of equal vorticities exist in a liquid, each vortex will describe a circle whose centre is the middle point of the line joining them, with velocity $M/2c$, where $2c$ is the distance between them; and therefore each vortex will move as if there existed a stress in the nature of a tension between them, of magnitude $M^2/4c^3$.[1]

To find the stream lines relative to the line joining the vortices, take moving axes, in which the axis of x coincides with the above-mentioned line; then

$$\psi = -\tfrac{1}{2} M \log \{y^2 + (x-c)^2\} \{y^2 + (x+c)^2\}.$$

Also

$$\dot{x} - \omega y = u = d\psi/dy,$$

$$\dot{y} + \omega x = v = -d\psi/dx,$$

where $\omega = M/2c^2$. Let

$$\chi = \psi + \tfrac{1}{2}\omega (x^2 + y^2),$$

therefore

$$\dot{x} = d\chi/dy, \quad \dot{y} = -d\chi/dx.$$

Multiplying by $\dot{y}$, $\dot{x}$ respectively, subtracting and integrating, we obtain

$$\chi = \text{const.} = A,$$

whence the equation of the relative stream lines is

$$\tfrac{1}{2}\omega (x^2 + y^2) - \tfrac{1}{2} M \log \{y^2 + (x-c)^2\} \{y^2 + (x+c)^2\} = A.$$

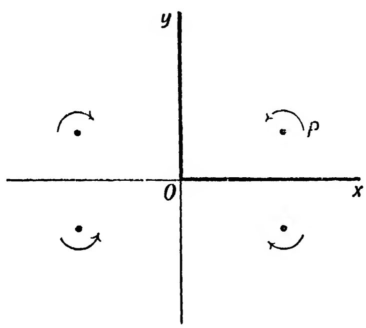
109. If two opposite vortices of vorticities m and $-m$ are present in the liquid, the vortices will move perpendicularly to the line joining them with velocity $M/2c$, where $2c$ is the distance between them.

[1] Greenhill, "Plane Vortex Motion," *Quart. Journ.* vol. xv. p. 20.

In this case there is evidently no flux across the plane which bisects the line joining the vortices, and which is perpendicular to it; we may therefore remove one of the vortices and substitute this plane for it. Hence a vortex in a liquid which is bounded by a fixed plane will move parallel to the plane, and the motion of the liquid will be the same as would be caused by the original vortex, together with another vortex of equal and opposite vorticity, which is at an equal distance and on the opposite side of the plane.

This vortex is evidently the image of the original vortex, and we may therefore apply the theory of images in considering the motion of vortices in a liquid bounded by planes.

110. If there is a vortex at the point (x, y) moving in a square corner bounded by the planes Ox, Oy, the images will consist of two negative vortices at the points $(-x, y)$, $(x, -y)$, and a positive vortex at the point $(-x, -y)$; for if these vortices be substituted for the planes, their combined effect will be to cause no flux across them.



Since the vortex is incapable of producing any motion of translation upon itself, its motion will be due solely to that produced by the combined effect of its images; whence,

$$\dot{x} = \frac{M}{2y} - \frac{My}{2(x^2+y^2)} = \frac{Mx^2}{2y(x^2+y^2)},$$

$$\dot{y} = -\frac{M}{2x} + \frac{Mx}{2(x^2+y^2)} = -\frac{My^2}{2x(x^2+y^2)};$$

therefore $$\dot{x}/x^3 + \dot{y}/y^3 = 0,$$

whence $$a^2(x^2+y^2) = x^2y^2,$$

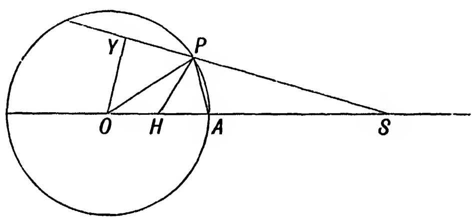
or $$r \sin 2\theta = 2a,$$

which is the curve described by the vortices. The curve in question is the reciprocal polar with respect to its centre of a four-cusped hypocycloid; but it also belongs to the class of curves called Cotes' Spirals, which are the curves described by a particle under the action of a central force varying inversely as the cube of the distance. Since

$$x\dot{y} - \dot{x}y = -\tfrac{1}{2}M$$

the vortex describes the spiral in exactly the same way as a particle would describe it, if repelled from the origin with a force $3M^2/4r^3$.

111. The method of images may also be applied to determine the current function due to a vortex in a liquid, which is bounded externally or internally by a circular cylinder.



Let H be the vortex, a the radius of the cylinder, $OH = c$; and let S be a point such that $OS = f = a^2/c$, then the triangles SOP and POH are similar, therefore

$$SPO = OHP,$$

$$OPH = OSP,$$

also
$$OSP + SPA = OAP = OPA$$
$$= OPH + HPA,$$

therefore
$$SPA = HPA.$$

Let us place another vortex of equal and opposite vorticity at S, then the velocity along OP due to the two vortices is

$$u = -\frac{M}{HP}\sin HPO + \frac{M}{SP}\sin SPO.$$

But
$$\frac{\sin HPO}{\sin SPO} = \frac{\sin HPO}{\sin OHP}$$
$$= OH/a$$
$$= HP/SP,$$

hence $u = 0$ and there is no flux across the cylinder.

Hence the image of a vortex inside a cylinder is another vortex of equal and opposite vorticity situated on the line joining the vortex with the centre of the cylinder, and at a distance a^2/c from the centre, and the vortex will describe a circle about the centre with a velocity

$$M/SH = Mc/(a^2 - c^2).$$

The current function of the liquid at a point (r, θ) within the cylinder is

$$\psi = -M \log HP/SP$$

$$= -\tfrac{1}{2} M \log \frac{r^2 + c^2 - 2rc\cos\theta}{r^2 + f^2 - 2rf\cos\theta}.$$

When the vortex is situated outside the cylinder, the image consists of a vortex of equal and opposite vorticity at H, together with a vortex of equal vorticity at O. The latter vortex does not produce any alteration in the normal velocity at the surface of the cylinder, and its existence arises from the fact that the circulation round any closed curve which surrounds the cylinder must remain unaltered. The circulation due to vortex at H is $-2\pi M$, whilst that due to the vortex at O is $2\pi M$, so that the two circulations cancel one another.

√112. We have shown that the velocity potential due to a source is $m \log r$; hence if we have a combination of a source of strength m, and a vortex of vorticity m', the velocity potential due to the two is

$$\phi = m \log r + M \tan^{-1} y/x,$$

where $M = m'/\pi$. Whence

$$u = \frac{mx - My}{x^2 + y^2}, \quad v = \frac{my + Mx}{x^2 + y^2}.$$

An arrangement of this kind is called *Rankine's free spiral vortex.*

In order to find the stream lines let us transfer to polar coordinates, and we find

$$\frac{dr}{dt} = \frac{m}{r}, \quad r\frac{d\theta}{dt} = \frac{M}{r},$$

whence if $m/M = \alpha$, we obtain

$$r = A\epsilon^{\alpha\theta},$$

and therefore the stream lines are equiangular spirals.

113. We shall conclude this Chapter by proving three fundamental properties of vortex motion.

We have defined a vortex line to be a line whose direction coincides with the direction of the instantaneous axis of molecular rotation. If through every point of a small closed curve a series of vortex lines be drawn, they will enclose a volume of fluid which may be called a vortex filament, or shortly a vortex.

We have shown that if the forces which act on the fluid have a potential, and the density is a function of the pressure, the motion of the fluid constituting the vortex can never become irrotational. It will now be shown that every vortex possesses the following three fundamental properties:

(i) *Every vortex is always composed of the same elements of fluid.*

(ii) *The product of the molecular rotation of any vortex into its cross section is constant with respect to the time, and is the same throughout its length.*

(iii) *Every vortex must either form a closed curve, or have its extremities in the boundaries of the fluid.*

To prove the first proposition, let P and Q be any two adjacent points on a vortex, ω the molecular rotation at P. Then by the definition of a vortex line, PQ is the direction about which the rotation ω takes place.

Let P', Q' be the positions of P and Q at the end of an interval δt; then we have to show that $P'Q'$ is the instantaneous axis of rotation at P'.

Let x, y, z be the coordinates of P; u, v, w the velocities of the element of fluid which at time t is situated at P.

If $PQ = h$, the coordinates of Q are evidently

$$x + h\xi/\omega, \quad y + h\eta/\omega, \quad z + h\zeta/\omega;$$

also since $u = F(x, y, z, t)$, it follows that if u_1, v_1, w_1 be the velocities of Q,

$$\begin{aligned} u_1 &= F(x + h\xi/\omega, \; y + h\eta/\omega, \; z + h\zeta/\omega, \; t) \\ &= u + \frac{h}{\omega}\left(\xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz}\right), \\ &= u + \frac{h\rho}{\omega}\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20), \end{aligned}$$

by § 20.

The coordinates of P' are

$$x + u\delta t,\ y + v\delta t,\ z + w\delta t,$$

and those of Q' are

$$x + h\xi/\omega + u_1\,\delta t = x + u\delta t + \frac{h\rho}{\omega}\left\{\frac{\xi}{\rho} + \frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right)\delta t\right\}$$

$$= x + u\delta t + \frac{h\rho\xi'}{\omega\rho'}$$

by (20), where ρ' is the density, and ξ', η', ζ' are the components of molecular rotation at P'.

Hence if h' denote the length of $P'Q'$, and λ', μ', ν' its direction cosines, then

$$\lambda'h' = h\rho\xi'/\omega\rho',\ \ \mu'h' = h\rho\eta'/\omega\rho',\ \ \nu'h' = h\rho\zeta'/\omega\rho' \ldots\ldots(21),$$

whence $$\lambda'/\xi' = \mu'/\eta' = \nu'/\zeta',$$

which shows that $P'Q'$ is the instantaneous axis of rotation at P', and therefore $P'Q'$ is the element of the vortex line, which at time t occupied the position PQ. This proves the first theorem.

To prove the second theorem, square and add (21) and we obtain

$$h' = h\rho\omega'/\omega\rho'.$$

But since the mass of the element is constant

$$\rho h\sigma = \rho'h'\sigma',$$

whence $$\sigma\omega = \sigma'\omega',$$

which proves that $\sigma\omega$ is independent of the time.

Since $$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0,$$

it can be shown by integrating this expression by parts throughout the interior of any closed surface, as was done in proving Green's Theorem, that

$$\iint(l\xi + m\eta + n\zeta)\,dS = \iiint\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)dx\,dy\,dz = 0,$$

or $$\iint\omega\cos\epsilon\,dS = 0,$$

where ϵ is the angle between the axis of rotation and the normal to S drawn outwards.

Now if we choose S so as to coincide with the surface of any finite portion of a vortex of small section, together with its two

ends, $\cos\epsilon$ vanishes except at the two ends; and is equal to $+1$ at one end, and -1 at the other; hence

$$\omega_1 dS_1 - \omega_2 dS_2 = 0,$$

which proves the second part of (ii).

To prove the third theorem, we observe that if a vortex did not form a closed curve or have its extremities in the boundary, it would be possible to draw a closed surface cutting the vortex once only, and the surface integral would not vanish.

The first theorem and the first part of the second theorem depend on dynamical considerations; the second part of this theorem and the third theorem are kinematical.

√ 114. The second and third theorems enable us to prove that the vorticity of any mass of rotationally moving liquid is constant.

Let us consider the case in which the rotationally moving liquid consists of a number of vortices which are surrounded by irrotationally moving liquid; and let us first confine our attention to a particular vortex of the system.

Draw any surface cutting each filament of the vortex at right angles once only; then the vorticity of the vortex is

$$\tau = \iint \omega dS \qquad (22).$$

Now we have proved that ωdS is constant along a vortex filament and is independent of the time; hence the surface integral is an absolute constant. By a theorem due to Sir G. Stokes (which is proved in my larger treatise) this surface integral is equal to the line integral $\frac{1}{2}\int(udx + vdy + wdz)$ taken round any closed curve drawn in the irrotationally moving liquid, which embraces the vortex once only. Hence the vorticity of a closed vortex ring, or of a vortex which has its extremities in the boundaries of the liquid, is equal to half the circulation due to the vortex.

In the same manner it can be shown that the vorticity of any number of vortices is equal to half the sum of the circulations due to each vortex.

When the whole of the liquid is moving rotationally, the vorticity is determined in the same manner; but since it is impossible to draw any closed curve throughout whose length $udx + vdy + wdz$ is a perfect differential, the vorticity cannot be expressed in the form of a line integral: but its value must be determined by means of the surface integral (22).

EXAMPLES.

1. If the axis of a hollow vortex be the axis of z, measured vertically downwards, the plane of xy being the asymptotic plane to the free surface, and if ϖ be the atmospheric pressure: prove that the equation of the surface at which the pressure is $\varpi + g\rho a$ is

$$(x^2 + y^2)(z - a) = c^3,$$

where c is a constant.

2. Three rectilinear vortices of equal vorticities form the edges of an equilateral triangular prism. Prove that they will always form the three edges of an equal prism.

3. If n rectilinear vortices of equal vorticities be initially at the angles of a prism whose base is a regular polygon of n sides, show that they will always be so situated, and that each vortex will describe the circumscribed cylinder with velocity $k(n-1)/2a$ where k is the velocity due to each vortex at unit distance and a is the radius of the cylinder. Show also that the equation of the relative stream lines referred to the radius through a vortex as initial line is $r^{2n} - 2a^n r^n \cos n\theta - b^{2n} = 0$.

4. The space on one side of the concave branch of a rectangular hyperbolic cylinder is filled with liquid, and a rectilinear vortex exists in the liquid; prove that it moves in a cylinder having the same asymptotic planes as the boundary.

5. The motion of a liquid in two dimensions is such that the rotation ζ is constant; prove that the general functional equation of the stream lines is

$$\phi(y + \iota x) + \chi(y - \iota x) - \tfrac{1}{2}\zeta(x^2 + y^2) = c.$$

Prove that if the space between one branch of the hyperbola $x^2 - 3y^2 = a^2$ and the tangent to its vertex be filled with liquid, it will be possible for the liquid to move steadily with constant rotation, and find the form of the stream lines.

6. A mass of liquid, whose outer boundary is an infinitely long cylinder of radius b, is in a state of cyclic irrotational motion and is under the action of a uniform pressure Π over its external surface. Prove that there must be a concentric cylindrical hollow whose radius a is determined by the equation

$$8\pi^3 a^2 b^2 \Pi = M\kappa^2,$$

where M is the mass of unit length of the liquid, and κ is the circulation.

If the cylinder receive a small symmetrical displacement, prove that the time of a small oscillation is

$$\frac{4}{\kappa}\pi^2 a^2 b^2 \sqrt{\frac{\log b/a}{b^4 - a^4}}.$$

7. Four straight vortices with alternately positive and negative rotations are placed symmetrically within a cylinder filled with liquid; prove that if the motion is steady, the distance of each vortex from the axis of the cylinder is nearly three-fifths of the radius of the latter.

8. Prove that three infinitely long straight cylindrical vortices of equal vorticities will be in stable steady motion when situated at the vertices of an equilateral triangle whose sides are large compared with the radii of the sections of the vortices; and that if they are slightly displaced, prove that the time of a small oscillation is the same as that of the time of revolution of the system in its undisturbed state.

9. A straight cylindrical vortex of uniform rotation ζ is surrounded by an infinite quantity of liquid moving irrotationally which is at rest at infinity; prove that the difference between the kinetic energy included between two planes at right angles to the axis of the cylinder and separated by unit distance when the cross section is an ellipse, and when it is a circle of equal area A, is

$$\rho\pi^{-1}\zeta^2 A^2 \log (a+b)/2\sqrt{(ab)},$$

where ρ is the density of the liquid, and a and b are the semiaxes of the ellipse.

10. A quantity of liquid whose rotation is uniform and equal to ζ, and whose external surface is a circular cylinder, surrounds a concentric cylinder of radius a. The external surface is subjected to a constant pressure Π. Prove that if the inner cylinder be removed, the velocity of the internal surface when its radius is α is equal to

$$\frac{1}{\alpha}\sqrt{\frac{(a^2-\alpha^2)(\zeta^2 c^2 - 2\Pi/\rho)}{\log \alpha^2/(\alpha^2+c^2)}},$$

where $\pi\rho c^2$ is the mass of the liquid per unit of length.

11. If a vortex is moving in a liquid bounded by a fixed plane, prove that á stream line can never coincide with a line of constant pressure.

12. If a pair of equal and opposite vortices are situated inside or outside a circular cylinder of radius a, prove that the equation of the curve described by each vortex is

$$(r^2 - a^2)^2 (r^2 \sin^2 \theta - b^2) = 4a^2b^2r^2 \sin^2 \theta,$$

where b is a constant.

PART II.

THEORY OF SOUND.

CHAPTER VI.

INTRODUCTION[1].

115. SOUNDS may be divided into two classes, musical sounds or notes, and unmusical sounds or noises, and a little consideration is sufficient to show that the former is the simpler phenomenon of the two. If, for example, one of the keys of a pianoforte be struck, we obtain a musical note, whereas if all the notes are sounded together, the result is an unmusical sound or noise, in which the different notes cannot be distinguished. We thus see that a combination of a number of musical notes will produce a noise, but on the other hand no combination of noises is capable of producing a musical note.

The sensation of sound is produced by means of vibrations of the atmosphere, which are first communicated to the tympanum of the ear, and are afterwards transmitted by the auditory nerve to the brain. That sound is produced by aerial vibrations can be experimentally verified in a number of ways. Thus if a bell be placed under the receiver of an air pump, and the air is gradually exhausted, the sound produced by the bell becomes fainter and fainter, and at last ceases to be heard. If, again, a note is produced by striking an ordinary finger bowl, the latter will be thrown into a state of tremor or vibration, the existence of which can be perceived by cautiously touching the bowl with the fingers; and if the vibrations are stopped by pressing the bowl between the hands, the note ceases upon the stoppage of the vibrations. In this case, the vibrations of the bowl are communicated to the atmosphere, and waves are propagated through the latter in all directions from the bowl.

[1] This and the following Chapter have been principally derived from vol. I. of Lord Rayleigh's Treatise.

116. At the commencement of Chapter IV. we explained the kinematics of wave motion, and we showed that the properties of a wave depend upon three quantities, viz. its *amplitude*, its *velocity of propagation* V, and its *wave length* λ. We may also, if we please, introduce the period τ instead of the wave length, since $V\tau = \lambda$. Notes also have three characteristics, viz. *intensity*, *pitch*, and a quality called *timbre*; and we must now enquire how these three physical characteristics of a note are connected with the geometrical constants of a wave.

117. In the first place it can be shown that the velocity of all notes in air and gases is very nearly the same[1]; for if this were not the case, a piece of music which is played in tune would become hopelessly discordant when heard by an observer situated at a considerable distance. A similar proposition is true of all substances which are capable of propagating sound; although the magnitude of the velocity of propagation depends upon the particular substance, being greater in the case of solids and liquids than in gases. Thus in dry air the velocity of sound at 0° C. is about 332 metres (i.e. 1089 feet) per second, whilst in water at 8° C. it is about 1435 metres (i.e. 4708 feet) per second[2]. It therefore follows that the properties of a note do not depend upon its velocity of propagation.

118. The intensity of sound is measured by the rate at which energy is propagated across a given area parallel to the waves, and is proportional to the square of the amplitude.

119. The pitch of a note is the quality by which its place in the musical scale is recognised. Thus the middle c of a pianoforte is said to have a pitch an octave lower than the next succeeding c in the scale. We shall now show that the pitch of a note depends upon the *frequency* of vibration, which has in Chapter IV. been defined to be the number of vibrations executed per second; and that the pitch rises as the frequency increases.

This is most easily shown by means of an apparatus called the

[1] It appears however that violent sounds, such as are caused by explosions, travel with a higher velocity than sounds produced by notes. Experiments made by Krupp's firm at Essen, for the purpose of ascertaining the velocity with which the reports of heavy guns travel, showed that the velocity may amount to 2034 feet per second. See "Sound velocity applied to range finding," Captain G. G. Aston, *Proc. Roy. Artillery Inst.* April, 1890.

[2] To reduce metres to feet, multiply by 3·2809.

Siren, invented by Cagniard de la Tour. This instrument consists of a stiff circular disc, which is capable of revolving about an axis, and is pierced with one or more sets of holes arranged at equal intervals around its circumference. A wind pipe in connection with bellows is presented perpendicularly to one of the holes, and the disc is made to revolve. When the time of revolution is small, the wind escapes by means of a succession of puffs; but after the time of revolution has sufficiently increased, the puffs blend into a single note of definite pitch; and if the time of revolution is still further increased, the pitch of the note rises in the scale. *This shows that the pitch of a note depends upon the frequency.* Another point of importance is that if the time of revolution is doubled, the two notes stand to one another in the relation of octaves; so that if f be the frequency of any particular note, the frequency of the note an octave higher is $2f$.

If $\epsilon^{\iota nt}$ be the time factor of a vibration, the frequency is $n/2\pi$; but since $n = 2\pi V/\lambda$, the frequency is also equal to V/λ; we have also shown that the pitch is independent of V, and it therefore follows that the frequency varies inversely as the wave length; consequently the shorter the wave length, the higher the pitch of the note.

The frequency of a given note is to a slight extent arbitrary, inasmuch as the ear is incapable of distinguishing slight differences of pitch. At the Stuttgart conference in 1834, it was recommended that the middle c of a pianoforte, which is written c', should correspond to 264 vibrations per second. The pitch usually adopted by acoustical instrument makers is taken to be $c' = 256$ or 2^8 vibrations per second, so that the frequencies of the octaves and sub-octaves are represented by powers of 2. Hence the wave length of c' is about 4·2 feet.

Trained ears are capable of recognising an enormous number of gradations of pitch, but inasmuch as the power of perception varies with different ears, it is somewhat difficult to assign limits to the audibility of notes. It is probable that the perception of pitch begins[1] when the number of vibrations in a second lies between 8 and 32, and ceases before it amounts to 40000.

120. All notes which are produced by musical instruments are of a highly compound nature; and when we discuss the

[1] Donkin's *Acoustics*, § 19.

dynamical part of the subject, it will be shown that the vibrations which are capable of being produced by a vibrating body are usually represented by an infinite series of terms of the form $A\epsilon^{\iota nt}$, in which the frequency of each term is different. A note which the ear is incapable of resolving is called by Helmholtz a *tone*. We thus see that a note which is represented by a series of terms of the type $A\epsilon^{\iota nt}$ is a compound note consisting of a number of tones, which are represented by the different terms of the series. The component tone of this series, whose frequency is the least, is called the *gravest* or *fundamental* tone, and the other tones are called overtones. It frequently happens (although there are exceptions), that the amplitudes of the component tones diminish as their pitches rise, so that the amplitude of the gravest tone is sufficiently large to impress its character upon the whole vibration; and in many cases is the note which is most distinctly heard. Lord Rayleigh states[1] that he has recently examined a large metal bell weighing about 3 cwt., and that the following tones could be plainly heard[2], viz.

$$e\flat,\quad f'\sharp,\quad e'',\quad b''.$$

The gravest tone $e\flat$ had a long duration. When the bell was struck by a *hard* body, the higher tones were at first predominant, but after a time they died away leaving $e\flat$ in possession of the field. When the striking body was soft, the original preponderance of the higher elements was less marked.

121. The word *timbre* is used to express a quality by which notes of the same intensity and pitch are distinguishable from one another, and which depends upon the nature of the instrument employed in producing the note.

122. Another phenomenon which we must notice is that of *beats*. Let us suppose for simplicity that two notes of the same amplitude and phase have slightly different frequencies m and n. The vibration produced by the combination of these two notes may be represented by the equation,

$$\begin{aligned} y &= a \cos 2\pi mt + a \cos 2\pi nt \\ &= 2 \cos \pi (m-n) t \cos \pi (m+n) t \\ &= 2 \cos \pi (m-n) t \cos \{2\pi m - \pi (m-n)\} t. \end{aligned}$$

[1] On Bells, *Phil. Mag.* Jan. 1890.

[2] c is the octave below, and c'' is the octave above the middle c of the pianoforte.

Since $m - n$ is small, the resultant vibration may be regarded as one whose amplitude and phase vary slowly with the time. We thus see that the amplitude vanishes whenever

$$t = (2s + 1)/2\,(m - n),$$

and is a maximum when $t = s/(m - n)$, where s is zero or any positive integer. Hence at intervals $(m - n)^{-1}$ there is absolute silence, and midway between the intervals of absolute silence, the intensity of the sound attains its maximum value. The intervals of silence are called *beats*, and the number of beats per second is $m - n$.

In order that beats may be heard distinctly, $m - n$, or the difference between the frequencies of the two notes, must be small.

CHAPTER VII.

VIBRATIONS OF STRINGS AND MEMBRANES.

123. If a piece of string or wire be tightly stretched between two fixed points, and be set in motion, either by being struck or rubbed with a bow, it is well known that a musical note will be produced. This arises from the circumstance that the string or wire is set into vibration, and we shall now proceed to investigate the theory of these vibrations.

If a thin metal wire, whose natural form is straight, is bent into a plane curve of any form, the resultant stresses across any normal section, due to the action of contiguous portions of the wire, consist of a tension, a shearing stress and a couple; and consequently in order to investigate the vibrations of instruments whose strings are made of wire, it would be necessary to construct a theory which would take these stresses into account. It is however obvious that although a thin string, made for example of catgut, is capable of sustaining a considerable tension, the resistance which it is capable of offering to shearing stress and to bending is very small in comparison with the resistance which it is capable of offering to stretching. We may therefore when dealing with strings made of catgut and similar materials, neglect the shearing stress and the couple, and may treat the string as *perfectly flexible.* We may define a perfectly flexible string to be a string which is incapable of offering any resistance to shearing stress or to bending. A string of this kind is an ideal substance which does not exist in nature; but inasmuch as most thin strings which

are not made of wire approximate to the condition of perfect flexibility, it will be desirable first of all to consider the vibrations of a perfectly flexible string. If however the string is too stiff to be treated as perfectly flexible, or is made of wire, the theory of the vibrations which it is capable of executing falls more properly under the head of the vibrations of bars. These will be considered in the next chapter.

The vibrations which a stretched string is capable of executing consist of two kinds, which may be treated as being independent of one another. The first kind consists of *transverse* vibrations, in which the displacement of every element is *perpendicular* (or very approximately so) to the undisplaced position of the string. The second class consists of *longitudinal* vibrations, in which the displacement is *parallel* to the undisplaced position of the string. It will thus be seen that, in the theory of transverse vibrations, the longitudinal displacement is supposed to be so small in comparison with the transverse displacement, that the former may be neglected in comparison with the latter.

Transverse Vibrations of Strings.

124. To find the equation of motion for transverse vibrations, it will be sufficient to consider the case in which the motion takes place in a plane. Let T_1 be the tension, ρ the linear density, i.e. the mass of a unit of length, y the displacement of the point whose abscissa is x, Y the impressed force per unit of mass.

If ϕ be the angle which any element δs makes with the axis of x, the equation of motion is

$$\rho \ddot{y} \delta s = \frac{d}{ds}(T_1 \sin\phi)\,\delta s + \rho Y \delta s.$$

Now $\sin\phi = dy/ds$; also since the displacement y is small, the curvature will also be small, and we may therefore put $ds = dx$. The tension T_1 may also be regarded as constant throughout the length of the string, whence the equation of motion becomes

$$\frac{d^2y}{dt^2} = \frac{T_1}{\rho}\frac{d^2y}{dx^2} + Y \quad\text{.......................}\quad (1).$$

If the motion does not take place in a plane, we may resolve the displacements and forces into two components respectively parallel to the axes of y and z, and we shall thus obtain a second

equation of the same form as (1), in which z, Z are written for y, Y respectively.

125. Let us now suppose that the length of the string is equal to l, and that there are no impressed forces; also let

$$a^2 = T_1/\rho \qquad (2).$$

Equation (1) now becomes

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2} \qquad (3).$$

To solve this equation, assume

$$y = F(x)\,\epsilon^{\iota mat}.$$

Substituting in (3) we obtain

$$\frac{d^2F}{dx^2} + m^2F = 0,$$

the solution of which is

$$F = C\sin mx + D\cos mx.$$

The solution of (3) may therefore be written in the form

$$y = \Sigma\,(C\sin mx + D\cos mx)\,\epsilon^{\iota mat} \qquad (4),$$

where m is at present undetermined, and C and D are complex constants.

The value of m will depend upon the particular problem under consideration. We shall now suppose that both ends of the string are fixed; in this case the conditions to be satisfied at the fixed ends are, that y and $\dot{y}$ should vanish when $x = 0$ and $x = l$. These conditions evidently require that

$$D = 0, \quad \sin ml = 0;$$

from the last of which we deduce

$$m = s\pi/l,$$

where s is a positive integer. Writing $C = A - \iota B$ and rejecting the imaginary part, the solution becomes

$$y = \Sigma_1^\infty\,(A_s\cos s\pi at/l + B_s\sin s\pi at/l)\sin s\pi x/l \qquad (5),$$

and therefore the period τ_s of the sth component is given by

$$\tau_s = \frac{2l}{sa} = \frac{2l}{s}\sqrt{\frac{\rho}{T_1}},$$

and the frequency

$$f_s = \frac{s}{2l}\sqrt{\frac{T_1}{\rho}}.$$

The gravest note corresponds to $s = 1$, and therefore its frequency is

$$f_1 = \frac{1}{2l}\sqrt{\frac{T_1}{\rho}}.$$

From these results we draw the following conclusions.

(i) The frequency is inversely proportional to the length; and therefore if the string be shortened, the pitch of the note will rise, and conversely if the string be lengthened, the pitch will fall. We thus see why it is that in playing a violin different notes can be obtained from the same string.

(ii) The frequency is proportional to the square root of the tension, accordingly if the string be tightened the pitch will rise.

(iii) The frequency is inversely proportional to the square root of the density; and therefore if two strings having the same lengths, cross sections and tensions, be made of catgut and metal respectively, the pitch of the note yielded by the catgut string will be higher than that yielded by the metal string; also the pitch of the note yielded by a thick string will be graver than that of the note yielded by a thin string, of the same material, length and tension.

If s be any integer other than unity, we learn from (5) that the displacement is zero at all points for which $x = rl/s$, where $r = 1, 2, 3, \ldots s - 1$; it therefore follows that, corresponding to the sth harmonic, there are $s - 1$ points situated at equal intervals along the string, at which there is no motion. These points are called *nodes*.

126. The constants A and B depend upon the initial circumstances of the motion. Now the motion of dynamical systems of which a string is an example may be produced either by displacing every point in any arbitrary manner, subject to the condition that the connections of the system are not violated; or by imparting to every point an arbitrary initial velocity, subject to the same condition. Hence the most general possible motion is obtained by communicating to every point of the string an initial displacement, and an initial velocity. We shall now show that when the initial displacements and velocities are given, the constants A and B are completely determined.

Let y_0, $\dot{y}_0$ be the initial displacements and velocities. Then it follows from (5) that

$$y_0 = \Sigma_1^\infty A_s \sin s\pi x/l \quad\ldots\ldots\ldots\ldots(6),$$

$$\dot{y}_0 = \Sigma_1^\infty (s\pi a/l) B_s \sin s\pi x/l \quad\ldots\ldots\ldots\ldots(7).$$

Now the integral $\int_0^l \sin(s\pi x/l)\sin(s'\pi x/l)\,dx$ is equal to zero if s and s' are different integers, and is equal to $\frac{1}{2}l$ if $s = s'$; whence multiplying (6) by $\sin s\pi x/l$ and integrating between the limits l and 0, we obtain

$$A_s = \frac{2}{l}\int_0^l y_0 \sin\frac{s\pi x}{l}\,dx \quad\ldots\ldots\ldots\ldots(8).$$

Similarly from (7)

$$B_s = \frac{2}{s\pi a}\int_0^l \dot{y}_0 \sin\frac{s\pi x}{l}\,dx \quad\ldots\ldots\ldots\ldots(9).$$

Since y_0, $\dot{y}_0$ are given functions of x, these equations completely determine the constants. We notice that B_s is zero when the initial velocity is zero, and that A_s is zero when there is no initial displacement.

127. As an example of these formulae, let us suppose that a point P, whose abscissa is b, of a string fixed at A and B, is displaced to a distance γ and then let go.

From $x = 0$ to $x = b$, $y_0 = \gamma x/b$, and therefore for this portion of the string

$$A_s' = \frac{2\gamma}{bl}\int_0^b x \sin\frac{s\pi x}{l}\,dx = \frac{2\gamma}{b}\left(-\frac{b}{s\pi}\cos\frac{s\pi b}{l} + \frac{l}{s^2\pi^2}\sin\frac{s\pi b}{l}\right).$$

From $x = b$ to $x = l$, $y_0 = \gamma(l - x)/(l - b)$, whence

$$A_s'' = \frac{2\gamma}{l(l-b)}\int_b^l (l - x)\sin\frac{s\pi x}{l}\,dx$$

$$= \frac{2\gamma}{b}\left\{\frac{b}{s\pi}\cos\frac{s\pi b}{l} + \frac{lb}{(l-b)s^2\pi^2}\sin\frac{s\pi b}{l}\right\}.$$

Whence adding we obtain

$$A_s' + A_s'' = A_s = \frac{2\gamma l^2}{s^2\pi^2 b(l-b)}\sin\frac{s\pi b}{l},$$

which determines A_s. This result shows that the amplitude of the gravest tone, which corresponds to $s = 1$, is greater than the amplitudes of any of the overtones. The gravest tone is therefore the most predominant.

128. The vibrations of a string which is set in motion by means of an initial velocity communicated *to every point of it*, may be investigated in a similar manner; but in many practical applications, a string is set in motion by means of an impulse *applied at some particular point*. The reader who desires to study the theory of the vibrations of a pianoforte wire, or of a violin string, is recommended to consult Donkin's *Acoustics*, and Lord Rayleigh's *Theory of Sound*. We shall confine ourselves to the simple case of the motion produced by an impulse F, applied at the point $x = b$.

Putting $n = s\pi a/l$, the value of y will be

$$y = \Sigma B_s \sin(nx/a) \sin nt \quad\text{.................}(10),$$

in which B_s has to be determined.

Now the work done by an impulse is equal to half the product of the impulse and the initial velocity of the point at which it is applied; and since the work done is equal to the kinetic energy of the initial motion, we immediately obtain the equation

$$F\dot{\eta}_0 = \rho\int_0^l \dot{y}_0^2 dx \quad\text{...... ....}(11),$$

where η is the value of y at the point at which the impulse is applied.

From (10), it follows that

$$F\dot{\eta}_0 = F\Sigma B_s n \sin nb/a,$$

and

$$\rho\int_0^l \dot{y}_0^2 dx = \rho\int_0^l \Sigma(B_s n \sin nx/a)^2 dx$$

$$= \tfrac{1}{2}\rho l \Sigma B_s^2 n^2;$$

and therefore restoring the value of n, (11) becomes

$$F\Sigma B_s s \sin s\pi b/l = \tfrac{1}{2}\rho\pi a\Sigma B_s^2 s^2 \quad\text{..............}(12).$$

Comparing both sides of this equation, we see that

$$B_s = \frac{2F}{\pi\rho a s}\sin\frac{s\pi b}{l},$$

and therefore

$$y = \frac{2F}{\pi\rho a}\Sigma\frac{1}{s}\sin\frac{s\pi b}{l}\sin\frac{s\pi x}{l}\sin\frac{s\pi a t}{l}.$$

At the nodes corresponding to the sth component, we have $x = rl/s$; and since in the preceding expression the sth component

of the displacement vanishes when $b=rl/s$, it follows that when the impulse is applied at a node, the corresponding component is absent.

129. We must now consider the motion of a string which is under the action of a periodic force $F(x)\cos pt$. It is well known that when an elastic body is set into vibration and left to itself, the motion gradually dies away and the system ultimately comes to rest. The reason of this is, that all such systems possess a property called viscosity or internal friction, by virtue of which the kinetic energy of the motion is gradually converted into heat. The effect of internal friction may be represented mathematically, by supposing that every element is retarded by a force proportional to its velocity, and we shall find it convenient in discussing motion due to a periodic force, to include the effect of viscosity.

It therefore follows that in (1) we must put

$$Y = F(x)\cos pt - k\dot{y},$$

where k is a constant, and the equation becomes

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt} = a^2\frac{d^2y}{dx^2} + F(x)\cos pt \qquad (13).$$

Since the motion which we are considering is periodic with respect to x as well as to t, we may assume

$$y = u\epsilon^{\iota mx}, \quad F(x) = E\epsilon^{\iota mx},$$

where $m = s\pi/l$; whence if $m^2a^2 = n^2$, (13) becomes

$$\frac{d^2u}{dt^2} + k\frac{du}{dt} + n^2u = E\cos pt \qquad (14).$$

The solution of this equation consists of two parts, viz. any particular solution of (14), together with the complementary function, which is the solution of the equation obtained by putting the right-hand side equal to zero, and which therefore contains two arbitrary constants. To find a particular solution, let us assume

$$u = c\cos(pt - e).$$

Substituting in (14), we obtain

$$c(n^2 - p^2)\cos(pt - e) - kpc\sin(pt - e) = E\cos e\cos(pt - e) - E\sin e\sin(pt - e);$$

whence equating coefficients of $\sin(pt - e)$, $\cos(pt - e)$, we obtain

$$c(n^2 - p^2) = E\cos e$$

$$cpk = E\sin e,$$

whence $$u = \frac{E \sin e}{pk} \cos (pt - e) \quad \text{..................(15)},$$

$$\tan e = \frac{pk}{n^2 - p^2} \quad \text{..............................(16)}.$$

In order to obtain the complementary function, let $u = \epsilon^{qt}$; substituting in (14) and putting $E = 0$, we obtain

$$q^2 + kq + n^2 = 0,$$

the roots of which are

$$q^2 = -\tfrac{1}{2}k \pm \iota\sqrt{(n^2 - \tfrac{1}{4}k^2)}.$$

Since k is always a small quantity, n will usually be greater than $\frac{1}{2}k$, in which case the complete solution of (14) may be written

$$u = A\epsilon^{-\frac{1}{2}kt} \cos \{\sqrt{(n^2 - \tfrac{1}{4}k^2)}\, t - \alpha\} + \frac{E \sin e}{pk} \cos (pt - e) \text{...(17)}.$$

130. The first term of this equation represents the *free* vibrations; that is to say the vibrations which the string is capable of executing, when it is set in motion in any manner and then left to itself. The period of these vibrations is $2\pi (n^2 - \frac{1}{4}k^2)^{-\frac{1}{2}}$, and the amplitude is proportional to $\epsilon^{-\frac{1}{2}kt}$; the free vibrations therefore diminish as the time increases and ultimately die away.

In dissipative systems, it is usual to express the effect of friction by means of a quantity called the *modulus of decay*; which is defined to be the time which must elapse before the amplitude has fallen to ϵ^{-1} of its original value. It therefore follows that if τ be the modulus of decay,

$$\tau = 2/k,$$

which determines the physical meaning of k. We thus see that if the friction is small, so that the amplitude diminishes very slowly with the time, τ must be large, and therefore k must be small.

In order to pass to the case of no friction, we must put $k = 0$, in which case the frequency is proportional to n. Hence one of the effects of friction is to lower the pitch of the free vibrations.

If $n < \frac{1}{2}k$, both values of q are real and the motion is non-periodic; but since both these values are negative, the free motion gradually dies away. It therefore follows that since the motion, whether periodic or not, always disappears after a sufficient time has elapsed, the expression for u ultimately reduces to the

last term, which represents the forced vibrations, and which we shall proceed to consider.

131. A *forced vibration* is a vibration produced and maintained by an external force. Its period is the same as that of the force, and it is consequently independent of the dimensions or constitution of the system. The amplitude of the forced vibration in the present case, when expressed in terms of n, p and k, is

$$\frac{E}{\{(n^2-p^2)^2+p^2k^2\}^{\frac{1}{2}}}.$$

If the system were absolutely devoid of friction, k would be zero, and the amplitude would be infinite when $n=p$. In practical applications k is usually small; and we thus obtain the important theorem, that *if a system is acted upon by a periodic force, whose period is equal, or nearly so, to one of the periods of the free vibrations of the system, the corresponding forced vibration will be large.*

This theorem can be illustrated in the case of stringed instruments; for if a note be sounded whose period is the same, or nearly the same, as that of the fundamental note of one of the strings, the string will often be heard to vibrate in unison with the note; whereas if the period of the note be different from that of any of the natural periods of the string, no sound will be heard.

132. When both extremities of the string are fixed, the general solution of (13) which of course includes (3) as a particular case, may be presented in the following form, which is frequently useful.

In this case $m=s\pi/l$, and therefore $n=s\pi a/l$, whence the complete solution may be written

$$y=\Sigma_1^\infty \phi_s \sin s\pi x/l \quad\text{.....................(18)},$$

where ϕ_s is a function of the time which satisfies (14), and whose value is therefore determined by (17). The quantities denoted by ϕ_s are called *normal* coordinates; and we shall now prove that the expressions for the kinetic and potential energies do not contain any of the products of the normal coordinates. This is the characteristic property of these quantities.

If T be the kinetic energy, we have

$$T=\tfrac{1}{2}\rho\int_0^l \dot{y}^2dx=\tfrac{1}{2}\rho\int_0^l \{\Sigma_1^\infty \dot{\phi}_s \sin s\pi x/l\}^2\,dx.$$

Since all the products vanish when integrated between the limits, we obtain

$$T = \tfrac{1}{4}\rho l \Sigma_1^\infty \dot{\phi}_s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

The potential energy is equal to the work done in displacing the string to its actual position. In order to calculate its value, let the string be held in equilibrium in its actual configuration at time t by means of a force Y applied at every point of its length. The value of this force per unit of mass is equal to

$$-\frac{T_1}{\rho}\frac{d^2y}{dx^2},$$

by (1). Let δV be the work which must be done by this force in order to displace every element of the string through a space δy; then the work done upon an element δs

$$Y\rho\delta s\delta y = -T_1\frac{d^2y}{dx^2}\delta s\delta y,$$

and therefore since $\delta s = \delta x$, the whole work done is

$$\delta V = -T_1\int_0^l \frac{d^2y}{dx^2}\delta y dx.$$

Integrating by parts, and recollecting that $\delta y = 0$ at both ends, we obtain

$$\delta V = T_1\int_0^l \frac{dy}{dx}\frac{d\delta y}{dx}dx = \tfrac{1}{2}T_1\int_0^l \delta\left(\frac{dy}{dx}\right)^2 dx,$$

whence

$$V = \tfrac{1}{2}T_1\int_0^l \left(\frac{dy}{dx}\right)^2 dx \ldots\ldots\ldots\ldots\ldots\ldots(20).$$

Substituting the value of y from (18) in (20), we obtain

$$V = \tfrac{1}{2}T_1\int_0^l \{\Sigma_1^\infty \phi_s(s\pi/l)\cos s\pi x/l\}^2 dx$$

$$= \frac{T_1\pi^2}{4l}\Sigma_1^\infty s^2\phi_s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

Longitudinal Vibrations of Strings.

133. We shall now obtain the equation of motion for the longitudinal vibrations of a string.

Let P and Q be two points whose abscissæ are x, $x + \delta x$; and let these points be displaced to P', Q'. If $x + \xi$ be the abscissa of P', the abscissa of Q' will be $x + \xi + (1 + d\xi/dx)\delta x$.

If T be the tension at any point,

$$T = E\frac{P'Q' - PQ}{PQ}$$

where E is Hooke's modulus of elasticity, whence

$$T = E\frac{d\xi}{dx}.$$

The equation of motion is

$$\rho\delta x\frac{d^2\xi}{dt^2} = \frac{dT}{dx}\delta x + X\rho\delta x$$

and therefore becomes

$$\frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2} + X,$$

where $a^2 = E/\rho$, and X is the impressed force per unit of mass.

This equation is of the same form as the equation for transverse vibrations, and can be solved in a similar manner.

The conditions to be satisfied at a fixed end are, that the displacement and velocity must be zero throughout the motion; and therefore at a fixed end

$$\xi = 0, \quad \dot{\xi} = 0$$

for all values of t.

The condition to be satisfied at a free end is that $T = 0$; and therefore at a free end

$$\frac{d\xi}{dx} = 0,$$

for all values of t.

Transverse Vibrations of Membranes.

134. The theory of the vibrations of membranes, is a particular case of the theory of the vibrations of thin elastic plates and shells. In general the stresses across any section of a thin plate or shell consist of[1] (i) a tension T, (ii) a tangential shearing stress M, (iii) a normal shearing stress N, (iv) a flexural couple G, (v) a torsional couple H. If however the membrane is very thin and perfectly flexible, the stresses reduce to a tension T, which in the dynamical problem of small transverse vibrations, may be taken to be equal in all directions, and constant all over the membrane.

We shall now obtain the equation of motion of a plane membrane.

[1] *Proc. Lond. Math. Soc.* Vol. XXI. p. 33.

Let w be the transverse displacement of any point, the coordinates of whose undisplaced position are $(x, y, 0)$; also let ρ be the superficial density, i.e. the mass of a unit of area of the membrane. If δs, $\delta s'$ be the sides of any small element of the membrane, we may write δx, δy for these quantities; whence the equation of motion is

$$\rho\delta x\delta y\ddot{w} = \frac{d}{dx}\left(T\delta y\frac{dw}{dx}\right)\delta x + \frac{d}{dy}\left(T\delta x\frac{dw}{dy}\right)\delta x,$$

which becomes

$$\frac{d^2w}{dt^2} = c^2\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right) \qquad (22),$$

if $c^2 = T/\rho$.

135. If the boundary of the membrane consists of a rectangle, whose sides are the axes and the lines $x = a$, $y = b$, we may assume as a particular solution of (22),

$$w = A \sin m\pi x/a \sin n\pi y/b \cos pt \qquad (23),$$

where $$p^2 = c^2\pi^2(m^2/a^2 + n^2/b^2) \qquad (24),$$

m and n being any integers; for this expression satisfies (22) and also makes w vanish at the boundaries. Equation (24) determines the frequency of the different notes; and from (23) we see that the nodal lines (i.e. the lines of no motion) consist of a system of $n-1$ lines parallel to x, whose distances apart are b/n, together with $m-1$ lines parallel to y, whose distances apart are equal to a/m.

If the membrane be square, $a = b$, and (23) and (24) become

$$w = A \sin m\pi x/a \sin n\pi y/a \cos pt,$$

$$p = c\pi(m^2 + n^2)^{\frac{1}{2}}/a.$$

The gravest note is obtained by putting $m = n = 1$, and corresponding to this note there are no nodes.

In the next place we shall determine the nodal lines corresponding to vibrations whose frequency is $\frac{1}{2}c\sqrt{5}/a$.

Here $$\sqrt{5} = \sqrt{(m^2 + n^2)},$$

which requires that $m = 2$, $n = 1$ or $m = 1$, $n = 2$; and therefore the complete vibration corresponding to this period is,

$$w = (C \sin 2\pi x/a \sin \pi y/a + D \sin \pi x/a \sin 2\pi y/a) \cos pt.$$

In this expression C and D depend solely upon the initial circumstances of the motion, and may have any values whatever consistent with the boundary conditions. If however we suppose

that the initial conditions are such, that the ratio C/D has an assigned value, we may obtain a variety of special cases.

(i) Let $D=0$. The nodal system now consists of the line $x=\frac{1}{2}a$, which bisects the membrane.

(ii) Let $C=0$, and we have a nodal line $y=\frac{1}{2}a$, similarly bisecting the membrane.

(iii) Let $C=D$; then the value of w may be written

$$w=4C\sin\pi x/a\sin\pi y/a\cos\tfrac{1}{2}\pi(x+y)/a\cos\tfrac{1}{2}\pi(x-y)/a\cos pt.$$

This expression vanishes when,

$$x=a,\quad y=a,\quad x+y=a,\quad x-y=a.$$

The first and second equations correspond to the edges; the fourth must be rejected, because it does not represent a line drawn on the membrane; and the third represents one of the diagonals of the square.

Since a nodal line may be supposed to be rigidly fixed without interfering with the motion, the preceding solution determines the frequency of the gravest note of a right-angled isosceles triangle.

(iv) Let $C=-D$, and we shall find that the nodal line is $y=x$, which represents the other diagonal of the square.

For further examples in this branch of the subject, the reader is referred to Chapter IX. of Lord Rayleigh's treatise.

136. The motion of a circular membrane, which is the best representative of a drum, cannot be solved by elementary methods. The simplest case of all, is when the vibrations are symmetrical with respect to the centre, so that (22) becomes

$$\frac{d^2w}{dt^2}=c^2\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right);$$

and if we put $w=F(r)\epsilon^{\iota pct}$, the equation for F is

$$\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF}{dr}+p^2F=0.$$

This equation cannot be integrated in finite terms. The two solutions are called Bessel's functions[1], from the name of their

[1] During recent years the ungrammatical phrase *Bessel* functions has begun to creep into mathematical literature. The use of a proper noun as an adjective is a violation of one of the most elementary rules of grammar; and in cases where it is not possible to form the corresponding adjective without introducing a cumbrous and inelegant term, the genitive case of the proper noun ought always to be employed.

discoverer; and the investigation of their properties constitutes an important branch of analysis. Algebraic solutions may however be invented, by supposing that the density, and therefore c, is a function of r.

EXAMPLES.

1. A string of length $l+l'$, is stretched with tension T between two fixed points. The linear densities of the lengths l, l' are m, m' respectively; prove that the periods τ of transverse vibrations are given by

$$m'^{\frac{1}{2}} \tan (2\pi l m^{\frac{1}{2}}/\tau T^{\frac{1}{2}}) = m^{\frac{1}{2}} \tan (2\pi l' m'^{\frac{1}{2}}/\tau T^{\frac{1}{2}}).$$

2. Investigate the motion of a string of length l, which is initially at rest in a straight line, each extremity of which is subject to the same obligatory motion $y = k \sin mat$. Show that if a sufficient period be allowed to elapse for the natural vibrations to subside, the position of the nodes will be given by the equation

$$2mx = ml + (2i+1)\pi,$$

where i is any integer.

3. A uniform string in the form of a circle of radius a, rests on a smooth plane under a central repulsion, whose value at distance r is ga^n/r^n. Show that if the string be slightly displaced, so that it is initially at rest and in the form of the curve

$$r = a + \Sigma_1^\infty a_m \cos m\theta,$$

its form at any subsequent time t, will be determined by the equation

$$r = a + \Sigma_1^\infty a_m \cos m\theta \cos m \left\{\frac{g}{a}\left(\frac{m^2+n-2}{m^2+1}\right)\right\}^{\frac{1}{2}} t.$$

Discuss this result (i) when $m=1, n=1$, and (ii) when $n=3$.

4. Three strings OA, OB, OC of the same material but of different lengths, are united at O, and are kept tight by being fastened to fixed points A, B, C, the angles BOC, COA, AOB being denoted by α, β, γ. Show that the times of vibration of the

different notes sounded when O is free, are determined by the equation for T, viz.

$$(\sin\alpha)^{\frac{1}{2}}\cot\pi T_1/T + (\sin\beta)^{\frac{1}{2}}\cot\pi T_2/T + (\sin\gamma)^{\frac{1}{2}}\cot\pi T_3/T = 0,$$

where T_1, T_2, T_3 are the times of the gravest notes of OA, OB, OC, when O is fixed.

5. If a stretched string of length l be fastened to two equal masses M, controlled by springs of strength μ allowing transversal vibration, and be plucked at its middle point, prove that the frequency n of vibration will be given by

$$\rho a \tan n\pi l/a = \mu/2n\pi - 2n\pi M,$$

where ρ is the line density, and ρa^2 the tension of the string.

6. A heterogeneous membrane in the shape of a circular annulus, whose edges are fixed and inner and outer radii are b and c, and whose density is μ/r^2, where r is the distance from the centre, is stretched with a tension T, and is performing small symmetrical normal vibrations. Show that a possible motion is given by

$$w = \{A\sin(p\log r/b) + B\sin(p\log c/r)\}\sin(apt+\alpha),$$

where $n\pi = p\log c/b$, n is an integer, and $a^2 = T/\mu$.

7. The fixed boundary of a membrane is square, and the centre of the membrane is displaced perpendicularly through a small space k, the membrane being made to take the form of two portions of intersecting circular cylinders. Prove that the origin being at the centre of the square, the vibrations are given by the equation

$$w = \Sigma A_{n_r}\cos\gamma t \sin n\pi(x+a)/2a \sin r\pi(y+a)/2a,$$

where $$4a^2\gamma^2 = c^2\pi^2(n^2+r^2).$$

Prove that in this case n and r are odd integers, and that

$$A_{n_r} = \frac{128k}{\pi^4(n^2-r^2)^2}\left(\frac{n^2+r^2}{nr} - 2\sin\tfrac{1}{2}n\pi\sin\tfrac{1}{2}r\pi\right),$$

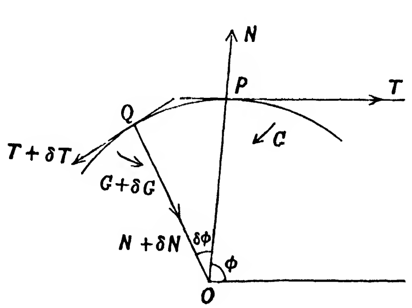
$$A_{n_n} = \frac{8k}{n^2\pi^2}\left(1+\frac{4}{n^2\pi^2}\right).$$

CHAPTER VIII.

FLEXION OF WIRES.

137. WE shall now investigate the theory of the equilibrium and the flexural vibrations of a thin rod or wire; and in the present chapter shall confine our attention to problems in two dimensions.

When the wire is not subjected to torsion, the stresses across any section, which are due to the action of contiguous portions of the wire, are completely specified by the following three quantities:—(i) a tension T perpendicular to the section, (ii) a normal shearing stress N, (iii) a flexural couple G. In the figure let PQ be a small element δs, ρ' the radius and O the centre of curvature at P after deformation, σ the density and ω the area of the cross section at P. Also let X, Y, L be the tangential and normal components of the impressed forces and the couple at P, per unit of mass, measured in the directions of T, N, G.



The equations of equilibrium of the wire, are obtained by resolving all the forces along the tangent and normal at P, and taking moments about this point; whence

$$T - (T + \delta T)\cos\delta\phi + (N + \delta N)\sin\delta\phi + \sigma\omega X\delta s = 0,$$
$$N - (T + \delta T)\sin\delta\phi - (N + \delta N)\cos\delta\phi + \sigma\omega Y\delta s = 0,$$
$$G - G - \delta G - (N + \delta N)\rho'\sin\delta\phi + \sigma\omega L\delta s = 0,$$

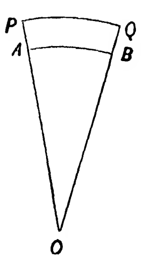
whence

$$\left.\begin{aligned} \frac{dT}{ds} - \frac{N}{\rho'} &= \sigma\omega X \\ \frac{dN}{ds} + \frac{T}{\rho'} &= \sigma\omega Y \\ \frac{dG}{ds} + N &= \sigma\omega L \end{aligned}\right\} \quad \text{.......................(1).}$$

138. We must now find an expression for the flexural couple G.

The curve which passes through the centre of inertia of each cross section, is called the *axis* of the wire. When a wire is bent in such a manner that its curvature is increased, the filaments into which the wire may be conceived to be divided, which lie on the outer side of the axis, will usually be extended; and those which lie on the inner side will usually be contracted, whilst the axis itself undergoes no extension nor contraction. Cases of course may occur, in which the axis undergoes extension or contraction, and when this is the case, the difficulties of the problem are greatly increased, and cannot be satisfactorily discussed without a knowledge of the Theory of Elasticity. We shall therefore confine our attention to the case in which the extension or contraction of the axis is so small (if it exists), that it may be neglected.

In the figure, let AB be the axis of the wire, PQ any filament, whose distance from AB is h, O the centre of curvature at B; also let these points after deformation be denoted by accented letters.



It can be proved in a variety of ways[1] that the tension T' at P' due to the action of contiguous portions of the wire is equal to the product of the extension of the element PQ and a physical constant called Young's modulus. If k is the resistance to compression, n the rigidity of the substance of which the wire is made and q

[1] See Thomson and Tait's *Natural Philosophy*, §§ 682—3.

Young's modulus, it is shown in treatises on Elasticity that

$$q = 9nk/(3k+n).$$

Hence if σ be the extension

$$T' = q\sigma \text{ ....(2).}$$

Now if ρ, ρ' be the radii of curvature before and after deformation

$$\frac{PQ}{AB} = \frac{\rho + h}{\rho}, \quad \frac{P'Q'}{A'B'} = \frac{\rho' + h}{\rho'}.$$

Since we assume that the extension of the axis may be neglected, $AB = A'B'$; whence neglecting h^2, &c.,

$$\sigma = \frac{P'Q' - PQ}{PQ} = h\left(\frac{1}{\rho'} - \frac{1}{\rho}\right),$$

and
$$T' = qh\left(\frac{1}{\rho'} - \frac{1}{\rho}\right).$$

Accordingly

$$G = \iint qh^2 \left(\frac{1}{\rho'} - \frac{1}{\rho}\right) dS$$

$$= q\kappa^2\omega \left(\frac{1}{\rho'} - \frac{1}{\rho}\right) \text{ ....(3),}$$

where $\kappa^2\omega$ is the moment of inertia of the cross section of the wire. Hence the flexural couple is proportional to the change of curvature.

The quantity of $q\kappa^2\omega$ is sometimes called the *flexural rigidity*, or the *coefficient of flexion.* We shall denote it by A.

In statical problems, the couple L will usually be zero, whilst the forces X, Y will be given; equations (1) together with (3) are therefore sufficient to determine the form of the wire.

139. The conditions to be satisfied at the ends of the wire are the following.

If the ends are subjected to constraining forces and couples, the values of the two stresses T and N at the ends must be respectively equal to the components along the tangent and normal of the constraining forces; and the couple G, must be equal to the constraining couple.

At a free end, T, N and G must vanish.

140. As an example of these formulæ, we shall consider the Elastica of James Bernoulli, which is the curve assumed by a naturally straight thin wire, whose ends are fastened together by a string of any given length.

Since there are no impressed forces, $X = Y = L = 0$; also $\rho' = ds/d\phi$, whence the first two of (1) become

$$\frac{dT}{d\phi} - N = 0, \quad \frac{dN}{d\phi} + T = 0,$$

and therefore

$$\frac{d^2T}{d\phi^2} + T = 0;$$

the integral of which is

$$T = C\cos\phi + D\sin\phi,$$

whence $$N = -C\sin\phi + D\cos\phi.$$

Let t be the tension of the string, and α and $\pi - \alpha$ the values of ϕ at the two extremities, then

$$-t\cos\alpha = N = -C\sin\alpha + D\cos\alpha,$$

$$-t\sin\alpha = T = C\cos\alpha + D\sin\alpha,$$

therefore $$C = 0, \quad D = -t.$$

Writing $A = q\kappa^2\omega$, and remembering that $\rho = \infty$, since the natural form of the wire is straight, we have $G = A/\rho'$, whence the third of (1) becomes

$$\frac{A}{\rho'}\frac{d}{d\phi}\left(\frac{1}{\rho'}\right) - t\cos\phi = 0 \quad\text{.....................(4).}$$

Integrating, we obtain

$$\frac{2A}{\rho'^2} - t\sin\phi = E.$$

Since $$G = 0 \text{ when } \phi = \alpha,\ E = -t\sin\alpha,$$

whence if $A/t = a^2$ $$\frac{ds}{d\phi} = \frac{a\sqrt{2}}{(\sin\phi - \sin\alpha)^{\frac{1}{2}}} \quad\text{.....................(5),}$$

which determines the intrinsic equation to the curve.

We may also integrate (4) in a different manner, for if the string be the axis of x, and its middle point be the origin,

$$\cos\phi = dy/ds,$$

and therefore (4) may be written

$$a^2 \frac{d}{ds}\left(\frac{1}{\rho'}\right) = t\frac{dy}{ds},$$

whence $$\rho' y = a^2;\text{..............................}(6),$$

no constant being required because $\rho'^{-1} = 0$ when $y = 0$.

Now $$\rho' = \frac{ds}{dy}\frac{dy}{d\phi} = \frac{dy}{d\phi}\sec\phi,$$

whence integrating (6) again

$$y^2 = 2a^2(\sin\phi - \sin\alpha)\text{.....................}(7).$$

The forms of the various curves which the wire is capable of assuming, are shown in Thomson and Tait's *Natural Philosophy*, Part II. p. 148.

If α lies between 0 and π, the maxima values of y are obtained by putting $\phi = \frac{1}{2}\pi$, and are therefore equal to $\pm 2a\sin(\frac{1}{4}\pi - \frac{1}{2}\alpha)$. The form of the curve is shown in the figures 1, 2 or 3 of that work; and if the curve be bent upon itself and the slight torsion be neglected, the forms are shown in figures 4 and 5. In all these cases except the first, in which the wire is bent into the shape of a bow, the maximum value of y is numerically equal to its minimum value. If, however, α lies between π and 2π, we may put it equal to $\pi + \beta$, in which case (7) becomes

$$y^2 = 2a^2(\sin\phi + \sin\beta).$$

In this case the maximum value of y occurs when $\phi = \frac{1}{2}\pi$, and is equal to $2a\cos(\frac{1}{4}\pi - \frac{1}{2}\beta)$, and the minimum when

$$\phi = 0, \text{ or } y = a(2\sin\beta)^{\frac{1}{2}};$$

the form of the curve is shown in fig. 7.

The constants a and α are capable of being determined when the lengths of the wire and string are given; and the equation of the curve in Cartesian coordinates can also be obtained, but to do this a knowledge of elliptic functions is required[1]. If, however, $\alpha = \frac{3}{2}\pi$, the integral in an algebraic form can be obtained; for since $\tan\phi = -dx/dy$, (7) becomes

$$-x = \int \frac{(y^2 - 2a^2)\,dy}{y(4x^2 - y^2)^{\frac{1}{2}}}$$
$$= -(4a^2 - y^2)^{\frac{1}{2}} + a\log\{2a/y + (4a^2/y^2 - 1)^{\frac{1}{2}}\} + C.$$

[1] Greenhill, *Mess. Math.* vol. VIII. p. 82.

Since $y = a\sqrt{2}$ when $x = 0$, $C = a\sqrt{2} - a\log(\sqrt{2}+1)$,

whence
$$x = (4a^2 - y^2)^{\frac{1}{2}} - a\sqrt{2} + a\log\frac{2a + (4a^2 - y^2)^{\frac{1}{2}}}{y(\sqrt{2}+1)}.$$

It will be noticed that this curve is the same as that described by an elliptic cylinder, in the limiting case between oscillation and rotation. See page 75.

141. Equation (5) enables us to prove a theorem discovered by Kirchhoff, and which is known as *Kirchhoff's kinetic analogue.* The theorem is, *that if a point move along the elastica with uniform velocity, the angular velocity of the tangent at that point, is the same as that of a pendulum under the action of gravity.*

If V be the velocity of the moving point, (5) may be written

$$\frac{d\phi}{dt} = \frac{V}{a\sqrt{2}}(\sin\phi - \sin\alpha)^{\frac{1}{2}}.$$

If we put
$$\chi = \tfrac{1}{2}\pi + \phi,\quad \alpha = \tfrac{1}{2}\pi + \beta,$$

this becomes
$$\frac{d\chi}{dt} = \frac{V}{a\sqrt{2}}(\cos\chi - \cos\beta)^{\frac{1}{2}},$$

which is the equation of motion of a common pendulum, whose length is equal $4ga^2/V^2$.

Stability under Thrust.

142. When a thin straight wire or column is subjected to a pressure or thrust, which is applied at one extremity in the direction of its length, experiment shows that as soon as the thrust exceeds a certain limit the wire commences to bend. There are several methods by which this limiting value can be obtained, but the following is perhaps the simplest.

Let a thrust P be applied at one extremity of the wire; and let P_0 be the limiting value of P which is just sufficient to produce bending. Then if P is less than P_0, no bending will take place; but if P is slightly greater than P_0, the wire will assume a sinuous form which differs very little from a straight line. We must therefore solve the equations of equilibrium on the hypothesis that the wire is not absolutely straight in its configuration of equilibrium, but assumes a slightly sinuous form; and we shall find that our solution leads to a certain equation of the form $P = \alpha$, where α is a quantity upon which the sinuous curve assumed

by the wire depends. Now P may have any value we please; if therefore we assign a value to P which is less than the least value of α, the equation $P=\alpha$ cannot be satisfied, which shows that equilibrium in the sinuous form is impossible. Hence the minimum value of α is the limiting value of P, which is just sufficient to produce bending.

143. Let ϖ denote the curvature of the wire when slightly bent; then $\varpi=\rho'^{-1}$, $\rho^{-1}=0$, $G=A\varpi$, and equations (1) become

$$\frac{dT}{ds}-N\varpi=0 \qquad (8).$$

$$\frac{dN}{ds}+T\varpi=0 \qquad (9),$$

$$A\frac{d\varpi}{ds}+N \ =0 \qquad (10).$$

From (8) and (10) we obtain

$$\frac{dT}{ds}+A\varpi\frac{d\varpi}{ds}=0,$$

whence

$$T=B-\tfrac{1}{2}A\varpi^2 \qquad (11).$$

Since ϖ is a very small quantity throughout the length of the wire, the constant B may be put equal to $-P$, where P is the thrust applied to the end of the wire; accordingly (11) may be written

$$T=-P-\tfrac{1}{2}A\varpi^2 \qquad (12).$$

Owing to the smallness of ϖ, we may neglect its cube; whence eliminating N from (9) and (10) we get

$$A\frac{d^2\varpi}{ds^2}+P\varpi=0,$$

the integral of which is

$$\varpi=C\cos\mu s+D\sin\mu s \qquad (13),$$

where $\mu^2=P/A$. We have now three cases to consider.

Case I. Let the lower end A of the wire be firmly clamped, whilst the upper end B is pressed vertically downwards by a force P, but is otherwise free; also let l be the length of the wire, and let the arc s be measured from A.

At the end B, G and therefore ϖ are zero; whence

$$C\cos\mu l + D\sin\mu l = 0 \qquad (14).$$

Also by considering the equilibrium of the whole wire, it follows that $N=0$ at A, whence by (10) $d\varpi/ds = 0$ when $s=0$, accordingly $D=0$. This requires that $\cos\mu l = 0$, whence

$$\mu l = (2n+1)\tfrac{1}{2}\pi,$$

which gives

$$P = \tfrac{1}{4}\pi^2(2n+1)^2 A/l^2 \qquad (15).$$

The least value of the right-hand side of (15) occurs when $n=0$; and this gives the thrust P which must be applied to the upper end of the wire to produce an infinitesimally small deflection; if, therefore, the thrust is less than this quantity, no deflection will take place and the wire will remain straight. Whence the condition of stability is that

$$P < \tfrac{1}{4}\pi^2 A/l^2 \qquad (16).$$

Case II. Let the wire be pressed between two parallel planes which are perpendicular to its undisplaced position. If the planes were perfectly hard, smooth and rigid (a condition which can only be approximately realized in nature), the ends of the wire would tend to slip on the slightest pressure being applied; we shall therefore suppose that the ends are in contact with mechanical appliances which will prevent any such slipping taking place, but are otherwise free.

Under these circumstances the terminal conditions are $\varpi = 0$ when $s=0$ and $s=l$. Whence by (13)

$$C=0, \quad \sin\mu l = 0, \quad \mu l = \pi,$$

and the condition of stability is that

$$P < \pi^2 A/l^2 \qquad (17).$$

Case III. Let both ends of the wire be clamped. The terminal conditions require that the values of N and ϖ at the two ends should be equal to one another. Consequently

$$D(1-\cos\mu l) = -C\sin\mu l,$$
$$C(1-\cos\mu l) = D\sin\mu l.$$

Eliminating C and D we obtain

$$\sin^2\tfrac{1}{2}\mu l = 0, \quad \tfrac{1}{2}\mu l = \pi,$$

and the condition of stability is

$$P < 4\pi^2 A/l^2 \qquad (18).$$

The value of ϖ may now be written

$$\varpi = C \cos 2\pi s/l,$$

which shows that there are two points of inflexion, which occur when $s = \frac{1}{4}l$ and $s = \frac{3}{4}l$.

The first case corresponds to a column or pillar whose lower end is cemented into a bed of concrete, whilst the upper end supports a building which simply rests upon but is not fastened to the pillar; and we see that in this case the weight required to cause the pillar to collapse is less than in the other two cases. The second case corresponds to a pillar or rod both of whose ends rest on bearings to which they are not cemented. The third case corresponds to a pillar whose ends are respectively cemented to the foundations and to the building supported. In the third case, the force required to cause the pillar to collapse is four times greater than in the second and sixteen times greater than in the first case.

144. Another interesting problem is the greatest height of a rod or column consistent with stability.

Let the lower end of a vertical rod be firmly fixed; then it can be shown by experiment that the rod will bend when its length exceeds a certain limit. To find this limit, we shall suppose that the rod when slightly bent is in equilibrium.

Since $dx = ds$, and $\rho^{-1} = \varpi$, equations (1) become

$$\left.\begin{aligned} \frac{dT}{dx} - N\varpi &= \sigma\omega g \\ \frac{dN}{dx} + T\varpi &= 0 \\ A\frac{d\varpi}{dx} + N &= 0 \end{aligned}\right\} \qquad (19).$$

From the first and third we obtain

$$T = B + \sigma\omega g x - \tfrac{1}{2}A\varpi^2,$$

where B is a constant. At the free end, where $x = l$, T and ϖ must vanish; whence

$$T = \sigma\omega g(x - l) - \tfrac{1}{2}A\varpi^2.$$

Substituting this value of T in the second of (19), eliminating N from the third and neglecting squares of ϖ, we obtain

$$A\frac{d^2\varpi}{dx^2} = \sigma\omega g(x - l) \qquad (20).$$

If W is the weight of the rod, $W=\sigma\omega gl$; also if we put

$$\varpi = z(l-x)^{\frac{1}{2}}, \quad r^2=\kappa^2(l-x)^3, \quad \kappa^2=4W/9Al,$$

(20) may be transformed into

$$\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}+\left(1-\frac{1}{9r^2}\right)z=0 \text{..............(21)},$$

the solution of which is

$$z=CJ_{\frac{1}{3}}(r)+DJ_{-\frac{1}{3}}(r),$$

where J is a Bessel's function.

When x is nearly equal to l, r is a very small quantity; accordingly if we integrate (21) by series and retain the two most important terms, the approximate value of z will be found to be

$$z=Cr^{\frac{1}{3}}+Dr^{-\frac{1}{3}},$$

whence $$\varpi = C\kappa^{\frac{1}{3}}(l-x)+D\kappa^{-\frac{1}{3}}.$$

Since $N=0$, when $x=l$, it follows that $d\varpi/dx$ must vanish when $x=l$; whence $C=0$, and

$$\varpi = D(l-x)^{-\frac{1}{2}}J_{-\frac{1}{3}}(r).$$

The value of ϖ at the free end of the rod is $Dl^{-\frac{1}{2}}J_{-\frac{1}{3}}(\kappa l^{\frac{3}{2}})$; but if the rod remains straight, $\varpi=0$ throughout its length. Whence if $\kappa l^{\frac{3}{2}}$ is less than the least root of the equation $J_{-\frac{1}{3}}(x)=0$, equilibrium in the bent form will be impossible, and the vertical form will be the only configuration of equilibrium. The least root of this equation is approximately equal to 1·88; whence the critical height at which the rod begins to bend is

$$l=2{\cdot}83\,(A/W)^{\frac{1}{3}}.$$

Further information relating to this subject will be found in the following papers[1].

[1] Greenhill, "On the greatest height consistent with stability," *Proc. Camb. Phil. Soc.* vol. IV. p. 65.

Ibid. "On the strength of shafting when exposed both to torsion and end thrust," *Proc. Inst. Mechan. Engineers*, Ap. 1883, p. 182.

Flexural Vibrations.

145. The preceding examples illustrate the use of these equations in statical problems; we must now proceed to consider the dynamical theory of the small vibrations of wires.

In order to obtain the equations of motion, we must write $X-\ddot{u}$, $Y-\ddot{v}$ for X and Y in the first two of (1), where u, v, are the tangential and normal displacements; and in the third equation we must write $L+\kappa^2\phi$ for L. The equations of motion are thus

$$\left.\begin{aligned}\frac{dT}{ds}-\frac{N}{\rho'}&=\sigma\omega(X-\ddot{u})\\ \frac{dN}{ds}+\frac{T}{\rho'}&=\sigma\omega(Y-\ddot{v})\\ \frac{dG}{ds}+N&=\sigma\omega(L+\kappa^2\ddot{\phi})\end{aligned}\right\}\text{..............} (22).$$

146. We shall now obtain the equation for determining the flexural vibrations of a wire, whose natural form is straight, when under the action of no forces.

In this case $\quad \dfrac{1}{\rho}=0,\ \dfrac{1}{\rho'}=-\dfrac{d^2v}{dx^2}.$

Since the curvature of the wire is small, ρ'^{-1} is a small quantity; hence *if there is no permanent tension*, the quotient T/ρ' is of the second order of small quantities, and may therefore be neglected; we may also write $-dx$ for ds (ds being measured in the figure in the opposite direction to dx), and the last two of (22) become

$$\frac{dN}{dx}=\sigma\omega\ddot{v}\text{........................}(23),$$

$$q\kappa^2\omega\frac{d^3v}{dx^3}+N=\sigma\kappa^2\omega\ddot{\phi}\text{ .... }(24).$$

Now $\quad \cot\phi=-dv/dx,$

and since ϕ is very nearly equal to $\frac{1}{2}\pi$,

$$\ddot{\phi}=d\ddot{v}/dx,$$

and therefore (24) becomes

$$q\kappa^2\omega\frac{d^3v}{dx^3}+N=\sigma\kappa^2\omega\frac{d^3v}{dt^2dx}\text{.................}(25);$$

whence eliminating N between (23) and (24) and putting $q/\sigma = b^2$, we obtain

$$\frac{d^2v}{dt^2} + \kappa^2 b^2 \frac{d^4v}{dx^4} - \kappa^2 \frac{d^4v}{dx^2 dt^2} = 0 \text{ .... } (26),$$

which is the required equation of motion.

147. The conditions to be satisfied at a free end are, that G and N should vanish there. It therefore follows from (3) and (25) that these conditions are

$$\frac{d^2v}{dx^2} = 0, \quad \frac{d^3v}{dt^2 dx} - b^2 \frac{d^3v}{dx^3} = 0 \text{ .... } (27).$$

These results agree with those given by Lord Rayleigh, *Theory of Sound*, vol. I. § 162; but the method employed in the text is different, and was suggested by a paper by Dr Besant[1].

In forming (26), it will be observed that we have not made use of the first of (22); so that the differential equation of motion, and also all the quantities upon which the vibrations depend, are expressed in terms of v. Now v is the displacement perpendicular to the length of the wire; hence such vibrations are frequently called *lateral* vibrations. The lateral vibrations of a wire, whose undisturbed form is a straight line, involve flexion and not extension. And it can also be shown that when the undisturbed form of the wire is any plane or tortuous curve, vibrations always exist which involve flexion and not extension; but since these vibrations usually involve the longitudinal displacement of the central axis, they are not *lateral* vibrations. Hence the vibrations, which we are considering, are more appropriately termed *flexural* vibrations.

148. The third term on the left-hand side of (26) is due to the *rotatory inertia* of the wire, i.e. to say, to the angular motion of the cross sections. This term is generally very small, and it may usually be neglected. When this is the case the equation of motion becomes

$$\frac{d^2v}{dt^2} + \kappa^2 b^2 \frac{d^4v}{dx^4} = 0 \text{ .... } (28),$$

[1] On the Equilibrium of a Bent Lamina, *Quart. Journ.* Vol. IV. p. 12.

When there is a permanent tension T_1, it will be found that we must write $q + T_1$ for q in (3); and the term T_1/ρ' in the second of (22), which becomes $-T_1 \omega d^2v/dx^2$, must be retained. We shall thus obtain the results given by Lord Rayleigh, § 188.

whilst the boundary conditions at a free end are

$$\frac{d^2v}{dx^2}=0,\quad \frac{d^3v}{dx^3}=0 \qquad\qquad (29).$$

149. For the complete discussion of these equations we must refer the reader to Chapter VIII. of Lord Rayleigh's treatise; but one or two special cases may be noticed.

If the wire is so long that it may be treated as infinite, we may neglect the conditions to be satisfied at its extremities. If therefore the vibrations consist of waves of length λ, we may assume as a solution of (28) that v is proportional to $\epsilon^{\iota pt+2\iota\pi x/\lambda}$. Substituting in (28) we obtain,

$$p^2 = 16\pi^4\kappa^2b^2/\lambda^4,$$

and therefore the frequency is

$$2\pi\kappa b/\lambda^2.$$

150. We shall now investigate the flexural vibrations of a wire[1] of length l.

Taking the origin at the middle point of the wire, we may assume

$$v = U \exp\,(\iota\kappa b m^2 t/l^2),$$

where U is a function of x, and m is a constant whose value has to be determined. Substituting in (28) we obtain

$$\frac{d^4U}{dx^4}=\frac{m^4U}{l^4}.$$

To solve this equation assume $U=\exp\,(pmx/l)$, and we see that the values of p are the four fourth roots of unity, viz. $1, -1, \iota, -\iota$. The solution may therefore be written

$$U = A\sin mx/l + B\sinh mx/l$$
$$+ C\cos mx/l + D\cosh mx/l \qquad\qquad (30).$$

151. We have now three cases to consider.

(i) Let both ends of the wire be free. The first of (29) requires that

$$-A\sin mx/l + B\sinh mx/l - C\cos mx/l + D\cosh mx/l = 0,$$

[1] Greenhill, *Mess. Math.* Vol. XVI. p. 115; Lord Rayleigh, *Theory of Sound*, Ch. VIII.

when $x = \pm \frac{1}{2}l$. This equation of condition may be satisfied in two different ways; we may first suppose that $C = D = 0$; and

$$-A \sin \tfrac{1}{2}m + B \sinh \tfrac{1}{2}m = 0 \quad\text{..............(31)},$$

or that $A = B = 0$, and

$$-C \cos \tfrac{1}{2}m + D \cosh \tfrac{1}{2}m = 0 \quad\text{..............(32)}.$$

The first solution corresponds to the first line of (30), which is an odd function of x, and may therefore be called odd vibrations; whilst the second solution corresponds to the second line of (30), which is an even function of x, and may be called even vibrations. We thus see that the odd and even vibrations are independent of one another.

Taking the case of the odd vibrations, the second of (29) requires that

$$-A \cos \tfrac{1}{2}m + B \cosh \tfrac{1}{2}m = 0,$$

and therefore by (31)

$$\tanh \tfrac{1}{2}m = \tan \tfrac{1}{2}m \quad\text{....................(33)}.$$

For the even vibrations the second of (29) gives

$$C \sin \tfrac{1}{2}m + D \sinh \tfrac{1}{2}m = 0,$$

and therefore by (32)

$$\tanh \tfrac{1}{2}m = -\tan \tfrac{1}{2}m \quad\text{.................(34)}.$$

Equations (33) and (34) determine the values of m for the odd and even vibrations respectively, and consequently the frequency of the different notes can be found.

(ii) Let both ends of the wire be clamped.

In this case the conditions to be satisfied at the ends are, that

$$U = 0, \quad dU/dx = 0 \quad\text{....................(35)},$$

the first of which expresses the condition that the displacement at each end should be zero, and the second that the direction of the axis should be unchanged.

The solution for this case may evidently be obtained by integrating the results of case (i) wire twice with respect to x, and consequently the values of m for the odd and even vibrations are given by (33) and (34).

(iii) Let the wire be clamped at $x = -\frac{1}{2}l$, and free at $x = \frac{1}{2}l$.

When $x = -\frac{1}{2}l$ equations (35) have to be satisfied; and when $x = \frac{1}{2}l$, the conditions are given by (29). Taking the value of U given by (30) and writing out the four equations of condition in full, it will be found that they can be satisfied in two ways, i.e. either, $B = C = 0$,

$$-A \sin \tfrac{1}{2}m + D \cosh \tfrac{1}{2}m = 0,$$

$$A \cos \tfrac{1}{2}m - D \sinh \tfrac{1}{2}m = 0,$$

which gives
$$\tanh \tfrac{1}{2}m = \cot \tfrac{1}{2}m \quad \text{.....................} \quad (36),$$

or, $A = D = 0$,

$$-B \sinh \tfrac{1}{2}m + C \cos \tfrac{1}{2}m = 0,$$

$$B \cosh \tfrac{1}{2}m + C \sin \tfrac{1}{2}m = 0,$$

which gives
$$\tanh \tfrac{1}{2}m = -\cot \tfrac{1}{2}m \quad \text{.....................} \quad (37).$$

Equations (33) and (34) are both included in the equation

$$\cos m \cosh m = 1 \quad \text{.....................} \quad (38),$$

and (36) and (37) in the equation

$$\cos m \cosh m = -1 \quad \text{.....................} \quad (39).$$

For a discussion of the roots of these equations, we must refer to Lord Rayleigh's *Theory of Sound*, Chapter VIII. and to Prof. Greenhill's paper.

Extensional Vibrations.

152. The equation of motion for extensional vibrations of a straight wire may be obtained immediately from the first of (22). In this case $\rho' = \infty$, $ds = -dx$, and therefore

$$\frac{dT}{dx} = \sigma\omega \frac{d^2u}{dt^2},$$

u being the longitudinal displacement.

But
$$T = q\omega \frac{du}{dx},$$

whence putting $q/\sigma = b^2$, we obtain

$$\frac{d^2u}{dx^2} = b^2 \frac{d^2u}{dt^2},$$

which is the same equation which we have obtained for the transverse vibrations of a string. The condition to be satisfied at a fixed end is that $u=0$; whilst the condition to be satisfied at a free end is that $T=0$, or $du/dx=0$.

In the case of a wire of infinite length, which propagates waves of length λ, we must put

$$u = \epsilon^{2\iota\pi x/\lambda + \iota pt},$$

and therefore $$p = 2\pi b/\lambda = \frac{2\pi}{\lambda}\sqrt{\frac{q}{\sigma}}.$$

In the corresponding case of flexural vibrations of the same wave-length,

$$p' = \frac{2\pi\kappa}{\lambda^2}\sqrt{\frac{q}{\sigma}},$$

whence $$p'/p = \kappa/\lambda.$$

Since λ is usually very much greater than κ, which is the radius of gyration of the cross section, we see that the pitch of notes arising from extensional vibrations is usually much higher than that of notes arising from flexural ones.

It will be observed that when the vibrations of a straight wire are extensional, the displacement is parallel to the length of the wire; hence such vibrations are sometimes called longitudinal vibrations. But if the natural form of the wire is curved, extensional vibrations usually involve the normal as well as the tangential displacement of the central axis.

EXAMPLES.

1. A naturally straight wire AB, of which the end A is fixed, is lying on a smooth horizontal plane, and the other end is pulled with a force F, whose direction is perpendicular to the undisplaced position of the wire. Prove that the projection of any length AP on the undisplaced position AB is equal to

$$(2F/A)^{\frac{1}{2}}\{\sqrt{(\cos\beta)} - \sqrt{(\cos\beta - \cos\phi)}\},$$

where ϕ is the angle which the normal at P makes with AB, and β is the value of ϕ at the end B.

2. If a uniform horizontal wire, both of whose ends are fixed, be displaced horizontally, so that one half is uniformly extended, and the other half is uniformly compressed, prove that the displacement at time t of any particle whose abscissa is x is

$$(4nl/\pi^2)\,\Sigma\,(2i+1)^{-2}\cos(2i+1)\,\pi at/2l\cos(2i+1)\,\pi x/2l,$$

where $2l$ is the length of the wire, the middle of which is the origin, and nl is the initial displacement of that point.

3. The extremities of a uniform wire of length l are attached to two fixed points distant l apart by springs of equal strength. Show that if the longitudinal displacement of the wire is represented by $P\epsilon^{\iota mat}\sin(mx/l+\alpha)$, the admissible values of m are given by the equation

$$(m^2q^2-l^2\mu^2)\tan m+2mql\mu=0,$$

where μ is the strength of either of the springs, and q the ratio of the tension to the extension in the wire.

4. An elastic wire, indefinitely extended in one direction, is firmly held in a clamp at the other end. If a series of simple transverse waves travelling along the wire be reflected at the clamp; show that the reflected waves will have the same amplitude as the incident waves, but that their phase is accelerated by one quarter of a wave-length.

5. A heavy wire of uniform section is carried on a series of supports in the same horizontal plane, L_r is the bending moment at the rth point of support, l_r the distance between the $(r-1)$th and the rth support, and m the mass of the wire per unit of length; prove that

$$L_{r-1}l_r+2L_r(l_r+l_{r+1})+L_{r+1}l_{r+1}=\tfrac{1}{4}mg\,(l_r^3+l_{r+1}^3).$$

6. Prove that if an elastic wire of length l, with flat ends, impinges directly with velocity V on a longer wire at rest, of length nl and of the same material and cross section, also with flat ends, the first wire will be reduced to rest by the impact; and the second wire will appear to move with successive advances of the ends with velocity V for intervals of time $2l/a$, and intervals of rest of $2(n-1)\,l/a$, a denoting the velocity of propagation of longitudinal vibrations.

7. An elastic rod of length l lies on a smooth plane, and is longitudinally compressed between two pegs at a distance l' apart. One peg is suddenly removed; prove that the rod leaves the other peg just as it reaches its natural state, and then proceeds with a velocity equal to $V(l-l')/l$, where V is the velocity of propagation of a longitudinal wave in the rod.

8. A metal rod fits freely in a tube of the same length, but of a different substance, and the extremities of each are united by equal perfectly rigid discs fitted symmetrically at the end. Show that the frequencies of the notes emissible, which have a node at the centre of the system, are given by $x/2\pi l$, where $2l$ is the length of the rod or tube, and n is a root of the equation

$$2Mx = ma \cot x/a + m'a' \cot x/a';$$

where M, m, m' are the masses of a disc, the wire, and the tube, and a, a' are the velocities of propagation of sound along the wire and the tube.

9. Two equal and similar elastic rods AC, BC are hinged at C so as to form a right angle, while their other extremities are clamped. One vibrates transversely and the other longitudinally; prove that the periods are $2l^2/f^2\theta^2$, where θ is given by the equation

$$1 + \cosh\theta\cos\theta + (\sin\theta\cosh\theta - \cos\theta\sinh\theta)(gl/f^2\theta)\cot(\theta^2 f^2/gl) = 0,$$

where l is the length of either rod, and f, g are two constants depending on the material.

10. The natural form of a thin rod when at rest is a circular arc, and the rod makes small oscillations about this form in its own plane. Assuming that the couple due to bending varies as the change of curvature, and that the tension follows Hooke's law, prove that if the arc be a complete circle the periods $2\pi/p$ are given by the quadratic,

$$p^4 - \{b(n^2+1) + an^2(n^2-1)\}p^2 + abn^2(n^2-1) = 0,$$

where n is any integer, and a, b are two constants which depend upon the moduli of stretching and bending, and on the radius of the circle.

11. If in the last example the arc be not a complete circle, but have both ends free and be inextensible, show that it can be made to vibrate symmetrically about its middle point by suitable initial conditions in a period $2\pi/p$, provided the angle 2θ, which the arc subtends at its centre, satisfies the equation

$$q(q^2+1)(q'^2-q''^2)\cot q\theta + q'(q'^2+1)(q''^2-q^2)\cot q'\theta$$
$$+ q''(q''^2+1)(q^2-q'^2)\cot q''\theta = 0,$$

where q^2, q'^2, q''^2 are the roots, real or imaginary, of the cubic

$$ax(x^2-1)^2 = (x+1)p^2.$$

CHAPTER IX.

THEORY OF CURVED WIRES.

153. WHEN a wire, whose natural form is straight or curved, is deformed in any manner, the stresses which act across any transverse section of the deformed wire are six in number, and consist of

T = a tension along the tangent to the central axis,

N_1 = a shearing stress along the principal normal,

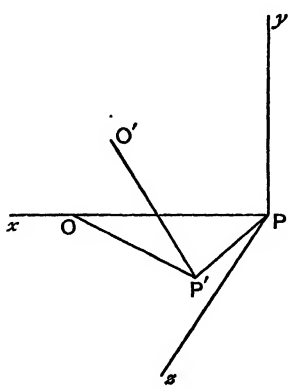
N_2 = a shearing stress along the binormal,

H = a torsional couple about the tangent,

G_1 = a flexural couple about the principal normal,

G_2 = a flexural couple about the binormal.

To obtain the equations of equilibrium[1], let P be any point on the central axis; Px, Py, Pz the principal normal, the binormal



[1] *Proc. Lond. Math. Soc.*, vol. XXIII. p. 105; *American Journal of Mathematics*, vol. XVII. p. 282.

and the tangent to the central axis at P. Let P' be any point on the central axis near P; O, O' the centres of principal curvature at P, P'; let $\delta\phi$, $\delta\eta$ be the angles of contingence and torsion at P, so that $POP' = \delta\phi$, $OP'O' = \delta\eta$; let ρ, σ be the radii of principal curvature and torsion at P. Also let $T + \delta T$, $N_1 + \delta N_1$, &c. be the values of the resultant stresses at P'; and $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$; $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ the components of the bodily forces and couples per unit of length of the wire.

The equations of equilibrium are obtained by resolving all the forces and couples which act upon PP' parallel to Px, Py, Pz. Resolving the forces parallel to Pz, we get

$$(T + \delta T)\cos\delta\phi - T - (N_1 + \delta N_1)\cos\delta\eta\sin\delta\phi$$
$$+ (N_2 + \delta N_2)\sin\delta\phi\sin\delta\eta + \mathfrak{Z}\delta s = 0,$$

or

$$\delta T - N_1\delta\phi + \mathfrak{Z}\delta s = 0.$$

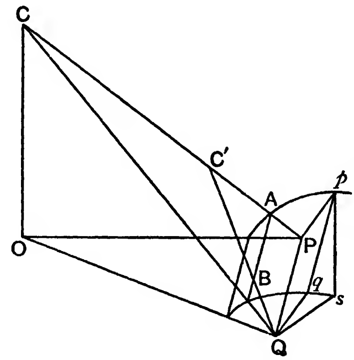
Treating all the other forces and couples in the same way, we obtain the following six equations:

$$\left.\begin{aligned}
\frac{dT}{ds} - \frac{N_1}{\rho} + \mathfrak{Z} &= 0\\
\frac{dN_1}{ds} - \frac{N_2}{\sigma} + \frac{T}{\rho} + \mathfrak{X} &= 0\\
\frac{dN_2}{ds} + \frac{N_1}{\sigma} + \mathfrak{Y} &= 0\\
\frac{dH}{ds} - \frac{G_1}{\rho} + \mathfrak{N} &= 0\\
\frac{dG_1}{ds} - \frac{G_2}{\sigma} + \frac{H}{\rho} - N_2 + \mathfrak{L} &= 0\\
\frac{dG_2}{ds} + \frac{G_1}{\sigma} + N_1 + \mathfrak{M} &= 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

In statical problems the three couples $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ are usually zero; but in dynamical problems they must be replaced by the time variations (taken with the negative sign) of the components of the angular momentum of the element.

154. We shall now find the values of the three couples, when the cross section of the wire is a circle, whose radius c is small compared with the radius of principal curvature; and we shall assume that the extension of the central axis may be neglected.

Let PQ be the central axis of the undeformed element; draw any plane through the tangent at P, let C be the centre of curvature of the element in this plane, and O the centre of principal curvature.



The flexion can obviously be effected by applying to the ends of the element equal and opposite couples, whose axes are perpendicular to the plane PCQ, which will be called the *plane of bending*. The effect of the bending will be to change the curvature at P in the plane PCQ, and also to extend or contract all filaments, such as pq, which are parallel to PQ, but the lines pq and PQ will remain sensibly parallel after bending.

The torsion can be effected by applying to the ends of the element equal and opposite couples, whose axes coincide with the tangents at P and Q. The effect of the torsion will be to twist all lines such as pq which lie on the surface of the cylinder $ApqB$ through small angles, so that after torsion they will assume positions slightly inclined to their former ones.

The flexion and torsion will also produce various deformations of a secondary character, but these may be neglected in working out the approximate solution which we shall obtain.

Let ρ, ρ' be the radii of curvature in the plane of bending before and after deformation, ρ_1 the radius of principal curvature; also let $qQO = \theta$, $qQC = \theta'$, $Qq = r$.

Before bending,

$$\frac{pq}{PQ} = \frac{\rho_1 - r\cos\theta}{\rho_1}.$$

The effect of the bending will be to displace the point q through a small space $r\,\delta\omega\cos\theta'$, where $\delta\omega$ is the rotation due to bending,

about a line through Q perpendicular to the plane of bending. Now if C' be the centre of curvature in the plane PCQ after bending,

$$\delta\omega = PC'Q - PCQ = PQ\left(\frac{1}{\rho'} - \frac{1}{\rho}\right),$$

whence the displacement of q along pq is

$$r\delta\omega \cos\theta' = PQ\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta',$$

consequently if σ_3' be the extension,

$$\sigma_3' = -\frac{r\delta\omega\cos\theta'}{pq} = -\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta' \quad\text{.........}(2),$$

if higher powers of r than the first be neglected.

The effect of the torsion will be to displace the line pq to the position ps, and therefore the above expression for the extension is not rigorously accurate when there is torsion as well as flexion, but the error depends upon the square of the small angle qQs and may be neglected.

If R' be the normal traction at q perpendicular to the cross-section, we have shown in § 138 that

$$R' = q\sigma_3' \quad\text{.........................}(3),$$

where q is Young's modulus; hence we obtain from (2) and (3),

$$R' = -q\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta'.$$

The flexural couple G about the normal through Q to the plane of bending is

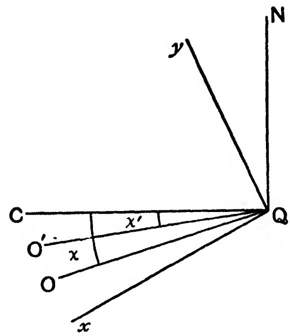
$$G = -\int_0^c \int_0^{2\pi} R' r^2 \cos^2\theta'\, dr\, d\theta'$$

$$= \tfrac{1}{4}\pi c^4 q\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) \quad\text{.........................}(4),$$

and is therefore proportional to the change of curvature in the plane of bending. The flexural couple about QC is obviously zero.

155. We shall now resolve this couple about two arbitrary axes Qx, Qy at right angles to one another in a plane perpendicular to the tangent at Q.

In the figure, QC is the normal to the wire in the plane of bending, QN is the normal to this plane at Q, and C is the centre



of curvature in this plane. Let $CQx = NQy = \phi$; then if G_x, G_y be the flexural couples about Qx, Qy, and A the flexural rigidity,

$$G_x = -G \sin\phi = -A\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\sin\phi \quad\text{...........(5)},$$

$$G_y = \quad G\cos\phi = \quad A\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\cos\phi \quad\text{...........(6)}.$$

Let QO, QO' be the principal normals at Q before and after bending; O, O' the centres of principal curvature; also let $CQO = \chi$, $CQO' = \chi'$. Let R_x, R_x', R_y, R_y' be the radii of curvature before and after bending in planes perpendicular to Qx, Qy, and let ρ_1, ρ_1' be the radii of principal curvature before and after bending. Then

$$\left.\begin{aligned} \frac{1}{\rho'} &= \frac{1}{\rho_1'}\cos\chi', & \frac{1}{\rho} &= \frac{1}{\rho_1}\cos\chi, \\ \frac{1}{R_x'} &= \frac{1}{\rho_1'}\sin(\phi - \chi'), & \frac{1}{R_x} &= \frac{1}{\rho_1}\sin(\phi-\chi), \\ \frac{1}{R_y'} &= \frac{1}{\rho_1'}\cos(\phi-\chi'), & \frac{1}{R_y} &= \frac{1}{\rho_1}\cos(\phi-\chi), \end{aligned}\right\}\text{......(7)}.$$

Since the curvature in the plane through the tangent which is perpendicular to the plane of bending is unchanged,

$$\frac{1}{\rho_1'}\sin\chi' = \frac{1}{\rho_1}\sin\chi \quad\text{......................(8)}.$$

From the first and second of (7) combined with (8) we get

$$\frac{1}{R_x'} = \frac{1}{\rho'} \sin\phi - \frac{1}{\rho_1} \cos\phi \sin\chi,$$

$$\frac{1}{R_x} = \frac{1}{\rho} \sin\phi - \frac{1}{\rho_1} \cos\phi \sin\chi,$$

whence
$$\frac{1}{R_x'} - \frac{1}{R_x} = \left(\frac{1}{\rho'} - \frac{1}{\rho}\right) \sin\phi,$$

accordingly
$$G_x = -A\left(\frac{1}{R_x'} - \frac{1}{R_x}\right) \quad\ldots\ldots\ldots\ldots (9),$$

and in the same way

$$G_y = \quad A\left(\frac{1}{R_y'} - \frac{1}{R_y}\right) \quad\ldots\ldots\ldots\ldots (10),$$

where $A = \frac{1}{4}\pi q c^4$ is the flexural rigidity. This shows that the flexural couples about the normals to any two planes at right angles to one another are proportional to the changes of curvature in those planes. The negative sign in (9) is accounted for by the fact that owing to the way in which the quantities are measured G_x is positive when the curvature is diminished.

156. We must now find the torsional couple.

The flexion simply displaces the point q along pq; the torsion produces a displacement along the circular arc to s, so that the line pq assumes the position ps.

Let the angles

$$qps = \psi, \quad qQs = \tau \,.\, PQ,$$

then
$$r\tau \,.\, PQ = pq \,.\, \psi,$$

whence
$$\psi = r\tau \,.\, \frac{PQ}{pq} = \frac{\rho r \tau}{\rho - r\cos\theta}.$$

Now ψ is the shearing strain perpendicular to Qq in the plane QqB, whence if H be the torsional couple,

$$H = n\int_0^c \int_0^{2\pi} \psi r^2 dr\, d\theta$$

$$= \tfrac{1}{2}\pi c^4 n\tau \quad\ldots\ldots\ldots\ldots (11).$$

The quantity τ is the change of twist, and is the same quantity which Mr Love denotes by $\tau' - \tau$.

Potential Energy.

157. Since the work done by a stress is equal to half the product of the stress into the strain produced, it follows that the work done by flexion is

$$\tfrac{1}{2}\int_0^c\int_0^{2\pi} R'\sigma_3' r\,dr\,d\theta = \tfrac{1}{2}q\left(\frac{1}{\rho'}-\frac{1}{\rho}\right)^2\int_0^c\int_0^{2\pi} r^3\cos^2\theta\,dr\,d\theta$$

$$= \tfrac{1}{8}\pi c^4 q\left(\frac{1}{\rho'}-\frac{1}{\rho}\right)^2$$

by (2) and (3).

The work done by torsion is

$$\tfrac{1}{2}n\int_0^c\int_0^{2\pi}\psi^2 r\,dr\,d\theta = \tfrac{1}{4}\pi c^4 n\tau^2.$$

It therefore follows that if W be the potential energy per unit of length,

$$W = \tfrac{1}{4}\pi c^4\left\{\tfrac{1}{2}q\left(\frac{1}{\rho'}-\frac{1}{\rho}\right)^2 + n\tau^2\right\} \quad\ldots\ldots\ldots\ldots\ldots(12).$$

Equilibrium of Naturally Straight Wires.

158. The preceding formulæ can be simplified when the wire is naturally straight. In this case the curvature in every plane through the axis of the wire is zero before deformation; and since the change of curvature in that plane through the tangent to the deformed wire which is perpendicular to the plane of bending is zero after deformation, it follows that the curvature in the above-mentioned plane is also zero after deformation. Hence the plane of bending is the osculating plane of the deformed wire.

From this it follows that

$$G_1 = 0, \quad G_2 = A/\rho,$$

whence, by the fourth of equations (1),

$$\frac{dH}{ds} = 0,$$

or

$$H = \text{const.},$$

which shows that the torsional couple is constant throughout the length of the wire.

This is a very important proposition.

159. We shall now proceed to integrate the equations of equilibrium of a naturally straight wire.

Since H is constant and $G_1 = 0$, it follows that if ϖ denote the curvature, so that $\varpi = 1/\rho$, equations (1) become

$$\frac{dT}{ds} - N_1\varpi = 0 \quad\text{.....................(13),}$$

$$\frac{dN_1}{ds} - \frac{N_2}{\sigma} + T\varpi = 0 \quad\text{.....................(14),}$$

$$\frac{dN_2}{ds} + \frac{N_1}{\sigma} = 0 \quad\text{.....................(15),}$$

$$\frac{G_2}{\sigma} - H\varpi + N_2 = 0 \quad\text{.....................(16),}$$

$$\frac{dG_2}{ds} + N_1 = 0 \quad\text{.....................(17).}$$

Since $G_2 = A\varpi$, we obtain from (13) and (17)

$$\frac{dT}{ds} + A\varpi\frac{d\varpi}{ds} = 0,$$

whence $$T = P - \tfrac{1}{2}A\varpi^2 \quad\text{.....................(18),}$$

where P is a constant.

From (16) we get $$N_2 = \left(H - \frac{A}{\sigma}\right)\varpi \quad\text{.....................(19),}$$

and from (13) and (18) $$N_1 = -A\frac{d\varpi}{ds} \quad\text{.....................(20).}$$

From (19) and (20) combined with (15) we get

$$H\varpi\frac{d\varpi}{ds} - A\frac{d}{ds}\left(\frac{\varpi^2}{\sigma}\right) = 0,$$

whence $$\frac{A}{\sigma} = \tfrac{1}{2}H + \frac{Q}{\varpi^2} \quad\text{.....................(21),}$$

where Q is a constant; accordingly by (19)

$$N_2 = \left(\tfrac{1}{2}H - \frac{Q}{\varpi^2}\right)\varpi \quad\text{.....................(22).}$$

To obtain a third integral, substitute the values of T, N_1, N_2 from (18), (20) and (22) in (14) and we get

$$A^2\frac{d^2\varpi}{ds^2} + (\tfrac{1}{4}H^2 - PA)\varpi - \frac{Q^2}{\varpi^3} + \tfrac{1}{2}A^2\varpi^3 = 0 \quad\text{.......(23).}$$

Integrating we obtain

$$\left(A\varpi\frac{d\varpi}{ds}\right)^2 = -\tfrac{1}{4}A^2\varpi^6 + (AP - \tfrac{1}{4}H^2)\,\varpi^4 + R\varpi^2 - Q^2 \;\text{.. (24)},$$

where R is another constant.

160. From (24) we see that $(d\varpi^2/ds)^2$ is a cubic function of ϖ^2, and therefore ϖ^2 can be expressed in terms of s by means of elliptic functions of the first kind. Let $\frac{1}{4}A^2Z$ denote this cubic function; then collecting our results from (18), (21) and (24), we have the following three first integrals of the equations of equilibrium, viz.

$$\left.\begin{aligned} T &= P - \tfrac{1}{2}A\varpi^2, \\ \frac{A}{\sigma} - \tfrac{1}{2}H &= \frac{Q}{\varpi^2}, \\ \frac{d\varpi^2}{ds} &= Z^{\frac{1}{2}}, \end{aligned}\right\} \quad \text{................ (25)}.$$

The first of (25) merely determines the tension, but the second leads to important results. If the curve assumed by the wire is a plane curve, $\sigma = \infty$; whence if Q is not zero, ϖ must be constant, and therefore the curve is a circle. If, however, Q is zero, σ is constant, and therefore the curve assumed by the wire is one of constant tortuosity; and if we suppose the curve to be plane, so that $\sigma = \infty$, it follows that H must be zero and the wire devoid of twist. From these results it follows that if a naturally straight wire is twisted as well as bent, the circle is the only plane curve which is a possible figure of equilibrium; but if the wire is bent without being twisted, a family of plane curves exist whose curvature is expressed in terms of the arc by means of the last of (25). These curves of course belong to the elastica family.

161. We shall now prove that if a naturally straight wire is twisted as well as bent, a helix is a possible form of equilibrium, provided proper forces and couples be applied to the ends of the wire.

Let α be the pitch of the helix, and a the radius of the cylinder on which the helix is traced, then

$$\frac{1}{\rho} = \frac{\cos^2\alpha}{a}, \quad \frac{1}{\sigma} = \frac{\sin\alpha\cos\alpha}{a} \quad \text{................ (26)},$$

also in the helix the principal normal is the normal to the cylinder.

Since the natural form of the wire is straight,

$$H = \text{const.} \quad G_1 = 0, \quad G_2 = A/\rho \;\ldots\ldots\ldots\ldots\; (27);$$

also none of the quantities can be functions of s, whence it follows from the general equations that

$$N_1 = 0, \quad T = \frac{H}{\sigma} - \frac{A}{\sigma^2}, \quad N_2 = \frac{H}{\rho} - \frac{A}{\rho\sigma} \;\ldots\ldots\ldots\; (28).$$

These equations combined with (26) give

$$\left.\begin{aligned} T &= \frac{H \sin\alpha \cos\alpha}{a} - \frac{A \sin^2\alpha \cos^2\alpha}{a^2}, \\ N_2 &= \frac{H \cos^2\alpha}{a} - \frac{A \sin\alpha \cos^3\alpha}{a^2}, \\ G_2 &= \frac{A \cos^2\alpha}{a}, \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\; (29),$$

whence

$$\left.\begin{aligned} T\cos\alpha - N_2 \sin\alpha &= 0, \\ T\sin\alpha + N_2\cos\alpha &= \frac{H}{a}\cos\alpha - \frac{A \sin\alpha\cos^2\alpha}{a^2} \end{aligned}\right\} \ldots\ldots (30).$$

Equations (30) show that the resultant force F which must be applied to the ends of the wire must be parallel to the axis of the cylinder on which the helix is traced, and that its magnitude is

$$F = \frac{H}{a}\cos\alpha - \frac{A}{a^2}\sin\alpha\cos^2\alpha \;\ldots\ldots\ldots\ldots\ldots\; (31).$$

The resultant couple $\mathfrak{G}$ is

$$\mathfrak{G}^2 = H^2 + \frac{A^2}{a^2}\cos^4\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\; (32).$$

The resultant force and couple are therefore to a certain extent arbitrary, since both contain the torsional couple H, the only limitation on whose value is that it must not be large enough to break the wire or to produce a permanent set. We have therefore two special cases to consider.

162. Case I. Let $H = 0$; then the terminal stresses consist of a pushing force or thrust P, whose value is

$$P = A/a^2 \,.\, \sin\alpha\cos^2\alpha,$$

together with a flexural couple G_2, whose value is $A/a \,.\, \cos^2\alpha$. The pitch of the helix is $\sin^{-1}(Pa/G_2)$, from which we see that in order that equilibrium may be possible Pa must not be greater than G_2.

Case II. Let $F=0$, then

$$H = \frac{A}{a} \sin\alpha\cos\alpha = \frac{A}{\sigma} \quad\text{.................(33)},$$

whilst $\mathfrak{G} = A/a \,.\, \cos\alpha$. The torsional couple is therefore proportional to the tortuosity; also since

$$H\cos\alpha - G_2\sin\alpha = 0,$$

$$H\sin\alpha + G_2\cos\alpha = A/a \,.\, \cos\alpha,$$

it follows that the terminal stress consists of a couple whose axis is parallel to the axis of the cylinder on which the helix is traced, and whose magnitude is $A/a \,.\, \cos\alpha$.

Stability.

163. In § 143 we investigated the stability of a naturally straight wire which was subjected to longitudinal thrust; we shall now investigate the stability when the wire is subjected to a torsional couple as well as a thrust.

In Case I., in which one end is unclamped, ϖ must vanish at the free end; and in Case II., in which both ends are unclamped, ϖ must vanish at each end. Now (21) may be written

$$\left(\frac{A}{\sigma} - \tfrac{1}{2}H\right)\varpi^2 = Q,$$

whence if ϖ vanishes anywhere, the constant Q must be zero. In Case III., both ends are clamped, and the tangents at the ends of the wire are consequently parallel; hence there must be at least two points of inflexion, and consequently ϖ must vanish at two points of the wire. Hence in this case also $Q=0$.

Writing $-P$ for P in (23), and recollecting that ϖ^3 is to be neglected, the equation becomes

$$\frac{d^2\varpi}{ds^2} + \mu^2\varpi = 0,$$

where $$\mu^2 = \frac{H^2}{4A^2} + \frac{P}{A} \quad\text{......................(34)},$$

whence $$\varpi = C\cos\mu s + D\sin\mu s,$$

accordingly if the right-hand side of (34) is less than the least value of μ, equilibrium in the sinuous form will be impossible, and the straight form will be stable.

Case I. Here one end is unclamped, whence $\varpi = 0$, when $s = l$; whilst at the clamped end $d\varpi/ds$, which is proportional to N_1, must also vanish; whence $D = 0$, and $\cos \mu l = 0$, therefore $\mu = \frac{1}{2}\pi/l$, and the condition becomes

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{\pi^2}{4l^2}.$$

Case II. Here $\varpi = 0$, when $s = 0$ and $s = l$, whence the condition is

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{\pi^2}{l^2}.$$

Case III. Here ϖ and $d\varpi/ds$ must be equal when $s = 0$ and $s = l$, and the condition is

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{4\pi^2}{l^2}.$$

All these results agree with our former ones, as can be seen by putting $H = 0$.

164. *A wire whose natural form is a tortuous curve is first unbent; secondly, the wire is twisted, and thirdly, the ends are joined together; it is required to find the condition that a circle may be a possible figure of equilibrium.*

When a circle is a figure of equilibrium none of the quantities can be functions of s; we therefore obtain from (1)

$$\left.\begin{matrix} T = 0 \quad , & N_1 = 0 \quad , & G_1 = 0 \quad , \\ H = \text{const.}, & G_2 = \text{const.}, & N_2 = H/a, \end{matrix}\right\} \quad \text{.........(35).}$$

The constancy of G_2 and H requires that the changes of curvature and twist which occur in passing from the natural to the circular form shall be constant quantities. These conditions will be satisfied if the natural form of the wire is a helix, which includes as a particular case a circular coil of *fine* wire, the radius of whose cross-section is small in comparison with the mean radius of the coil.

165. If in the last example the natural form of the wire is a straight line, the circular form will be unstable when the twist exceeds a certain limit. To find this limit, we observe that since the natural form is straight, it follows that in the circular form $G_2 = A/a$; whilst in the sinuous form H will still be constant and

G_1 zero. But T and N_1 will be small quantities depending on the change of curvature ϖ, also we must write

$$G_2 = A/a + A\varpi, \quad N_2 = H/a + N_2'.$$

Accordingly equations (1) become

$$\frac{dT}{ds} - \frac{N_1}{a} = 0,$$

$$\frac{dN_1}{ds} - \frac{H}{a\sigma} + \frac{T}{a} = 0,$$

$$\frac{dN_2'}{ds} = 0,$$

$$-\frac{A}{a\sigma} + H\varpi - N_2' = 0,$$

$$A\frac{d\varpi}{ds} + N_1 = 0.$$

From the first and fifth we get

$$T = -A\varpi/a.$$

Substitute this value of T in the second equation; then eliminate σ by means of the fourth, and differentiate the result with respect to s, taking account of the third, and we obtain

$$\frac{d^3\varpi}{ds^3} + \left(\frac{H^2}{A^2} + \frac{1}{a^2}\right)\frac{d\varpi}{ds} = 0,$$

whence $$\frac{d\varpi}{ds} = P\sin(\mu s + \alpha),$$

where $$\mu^2 = \frac{H^2}{A^2} + \frac{1}{a^2}.$$

Since $d\varpi/ds$ must be periodic when s is changed into $s + 2n\pi a$, it follows that $\mu = n/a$, whence

$$H^2a^2 = A(n^2 - 1).$$

Hence if H is less than the least value of the right-hand side, equilibrium in the sinuous form will be impossible, and the circular form will be stable. The displacement corresponding to $n = 1$, is a bodily displacement which is unaffected by the strain; hence the least value occurs when $n = 2$, and the condition of stability is that

$$Ha < A\sqrt{3},$$

or $$\tau < q\sqrt{3}/2na,$$

where τ is the twist.

This result appears to have been first given by Michell[1], who obtained it by supposing the wire to perform small oscillations. For a large number of metals, the ratio q/n is equal to about $2\frac{1}{2}$, in which case the total twist must be less than $2\pi \times 2.16$, that is less than eight and a half right angles.

Vibrations of a Circular Ring.

166. When the equilibrium of the wire in the last example is stable, it is capable of performing vibrations which yield a musical note. The period of these vibrations satisfies a cubic equation; but when there is no twist, the cubic breaks up into two factors whose roots are always real. One factor represents vibrations in which the central axis assumes a tortuous curve; whilst in the other type the motion takes place in the plane of the ring. The vibrations of the first type have been considered by myself[2]; whilst those of the second type have been discussed by Hoppe[3]. We shall now investigate the period equation of the latter type.

In this case, we may in equations (22) of § 145 put $ds = ad\phi$, and $\rho' = a$, since the difference between ρ'^{-1} and a^{-1} may be neglected when multiplied by T or N. These equations therefore become, measuring u in the opposite direction, so that u and ϕ increase together,

$$\left.\begin{aligned} \frac{dT}{d\phi} - N &= \sigma a \omega \ddot{u} \\ \frac{dN}{d\phi} + T &= -\sigma a \omega \ddot{v} \\ \frac{dG}{d\phi} + Na &= 0 \end{aligned}\right\} \text{.....................} (36),$$

the rotatory inertia being neglected.

We must now find an expression for the change of curvature due to deformation.

If R, Φ be the coordinates after displacement, of the point on the axis which was initially at (a, ϕ), then

$$R = a + v, \quad \Phi = \phi + u/a.$$

[1] *Mess. Math.* vol. XIX. p. 184.

[2] *Proc. Lond. Math. Soc.* vol. XXIII. p. 120; *American Journal*, vol. XVII. p. 315.

[3] *Crelle*, vol. LXIII.; Lord Rayleigh, *Theory of Sound*, § 233.

Hence if P be the perpendicular from the centre on to the tangent to the deformed axis, at the point in question, we have by a well-known formula,

$$\frac{1}{\rho'} = \frac{1}{R}\frac{dP}{dR}.$$

Now
$$\frac{1}{P^2} = \frac{1}{R^2}\left\{1 + \frac{1}{R^2}\left(\frac{dR}{d\Phi}\right)^2\right\}$$

$$= \frac{1}{R^2}\left\{1 + \frac{(dv/d\phi)^2}{R^2(1 + du/ad\phi)^2}\right\};$$

also the displacements and their differential coefficients are all small quantities; whence expanding and neglecting cubes of small quantities, the above equation becomes

$$P = a + v - \frac{1}{2a}\left(\frac{dv}{d\phi}\right)^2,$$

whence
$$dP = \left(1 - \frac{1}{a}\frac{d^2v}{d\phi^2}\right)\frac{dv}{d\phi}\,d\phi.$$

Also
$$dR = \frac{dv}{d\phi}\,d\phi,$$

therefore
$$\frac{1}{\rho'} - \frac{1}{a} = -\frac{1}{a^2}\left(\frac{d^2v}{d\phi^2} + v\right) \quad\text{..................(37)},$$

which determines the change of curvature in terms of the normal displacement.

We must next find the condition that the axis undergoes no extension.

The elementary arc ds' of the deformed surface is given by the equation

$$ds'^2 = (dv)^2 + (a + v)^2(d\phi + du/a)^2,$$

and since this is equal to $a^2d\phi^2$, we obtain, neglecting squares of small quantities,

$$\frac{du}{d\phi} + v = 0 \quad\text{..........................(38).}$$

which is the condition of inextensibility.

Substituting from (37) and (3) of § 138 in the last of (36) we obtain

$$\frac{q\kappa^2\omega}{a^3}\left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi}\right) = N.$$

From the first two of (36) we obtain

$$\frac{d}{d\phi}\left(\frac{d^2N}{d\phi^2}+N\right)=-\sigma a\omega\left(\frac{d^2\ddot{v}}{d\phi^2}+\frac{d\ddot{u}}{d\phi}\right)$$

$$=-\sigma a\omega\left(\frac{d^2\ddot{v}}{d\phi^2}-\ddot{v}\right),$$

by (38), whence eliminating N we obtain

$$\frac{q\kappa^2}{\sigma a^4}\left(\frac{d^2}{d\phi^2}+1\right)^2\frac{d^2v}{d\phi^2}+\left(\frac{d^2}{d\phi^2}-1\right)\ddot{v}=0 \quad\text{..........(39).}$$

To solve this equation, assume that $v \propto \epsilon^{\iota pt+\iota s\phi}$, and we obtain

$$p^2=\frac{q\kappa^2s^2(s^2-1)^2}{\sigma a^4(s^2+1)} \quad\text{..........................(40).}$$

If the wire is a complete circle, v must necessarily be periodic with respect to ϕ, and therefore s must be an integer, unity and zero excluded. We therefore see that there are an infinite number of modes of vibration, whose frequencies are obtained by putting $s=2, 3, 4 \ldots$ in (40).

If the wire is not a complete circle, s is not an integer; its values in terms of p are the six roots of (40), but since p is unknown, another equation is necessary. This equation is obtained by considering the boundary conditions to be satisfied at the free ends, which are that T, N and G should vanish there. These conditions will furnish six additional equations, by means of which the six constants which appear in the solution of (39) can be eliminated, and the resulting determinantal equation combined with (40), will determine the frequency[1].

[1] See Lamb, "On the flexure and the vibrations of a curved bar," *Proc. Lond. Math. Soc.* vol. XIX. p. 365.

CHAPTER X.

EQUATIONS OF MOTION OF A PERFECT GAS.

167. We have already called attention to the fact, that air is the vehicle by means of which sound is transmitted; we must therefore investigate the equations of motion of a gas.

The general equations of fluid motion, which we obtained in Chapter I, are of course applicable to elastic fluids such as air and other gases, as well as to incompressible fluids such as water; but in order to investigate the propagation of sound in gases, these equations require modification.

In all problems relating to vibrations, the velocities upon which the vibrations depend, are usually so small that their squares and products may be neglected; also the variation of the density of the gas is usually a small quantity. If therefore a gas, which is at rest, be disturbed by the passage of sound waves, we may write d/dt for $d/dt + ud/dx + vd/dy + wd/dz$, and also put $\rho = \rho_0 (1 + s)$, where s, which is called the condensation, is a small quantity. The equations of motion therefore become

$$\left.\begin{aligned} \frac{du}{dt} &= X - \frac{1}{\rho}\frac{dp}{dx} \\ \frac{dv}{dt} &= Y - \frac{1}{\rho}\frac{dp}{dy} \\ \frac{dw}{dt} &= Z - \frac{1}{\rho}\frac{dp}{dz} \end{aligned}\right\} \qquad \text{.....................(1)},$$

whilst the equation of continuity, § 6, equation (5), becomes

$$\frac{ds}{dt} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \qquad \text{..................(2)}.$$

We shall also suppose that the bodily forces (if any) which act upon the gas arise from a potential U, and also that the motion is irrotational; (2) therefore becomes

$$\frac{ds}{dt} + \nabla^2\phi = 0 \quad\text{..........................(3).}$$

We have already shown that when the motion is irrotational, the pressure is determined by the equation

$$\int \frac{dp}{\rho} + U + \frac{d\phi}{dt} + \tfrac{1}{2}q^2 = C \quad\text{..................(4).}$$

Now q^2 is to be neglected, also if we assume Boyle's law to hold, we shall have

$$p = k\rho = k\rho_0(1+s),$$

and therefore

$$\int \frac{dp}{\rho} = k\int \frac{ds}{1+s} = k\log(1+s) + C',$$

$$= ks + C',$$

neglecting s^2 &c. Whence (4) becomes

$$ks + U + \dot{\phi} + C' = C.$$

If there were no forces in action and no motion, the first three terms would be zero; whence $C' = C$, and therefore,

$$ks + U + \dot{\phi} = 0 \quad\text{........................(5),}$$

or if δp denote the small variable part of p, (5) may be written

$$\frac{\delta p}{\rho_0} + U + \dot{\phi} = 0 \quad\text{........................(6).}$$

Eliminating s between (3) and (5) we obtain

$$\frac{d^2\phi}{dt^2} = k\nabla^2\phi - \frac{dU}{dt} \quad\text{........................(7).}$$

Equation (6) and (7) are the fundamental equations of the small vibrations of a gas.

√168. In almost all the applications of these equations, no impressed forces act, and therefore $U = 0$; accordingly (7) becomes

$$\frac{d^2\phi}{dt^2} = k\nabla^2\phi \quad\text{..........................(8).}$$

Let us now suppose that plane waves of sound are propagated in a gas of unlimited extent. Let l, m, n be the direction cosines

of the wave front, a the velocity of propagation of the wave. We may assume

$$\phi = A\epsilon^{\iota\kappa(lx+my+nz-at)}.$$

Substituting in (8) we obtain

$$k = a^2 \quad\text{.............................(9).}$$

This equation determines the physical meaning of k, and shows that it is equal to the square of the velocity of propagation. We may therefore write (8) in the form

$$\frac{d^2\phi}{dt^2} = a^2\nabla^2\phi \quad\text{.......................(10).}$$

Let ξ, η, ζ be the displacements of an element of fluid, then

$$\frac{d\xi}{dt} = \frac{d\phi}{dx} = A\iota\kappa l\epsilon^{\iota\kappa(lx+my+nz-at)}$$

whence $$\xi = -(Al/a)\,\epsilon^{\iota\kappa(lx+my+nz-at)},$$

with similar expressions for η and ζ. We thus obtain

$$\xi/l = \eta/m = \zeta/n,$$

which shows that the displacement is *perpendicular* to the front of the wave. This constitutes one of the fundamental distinctions between waves of sound and waves of light, for it is well known that in a wave of light the direction of displacement always lies *in* the wave front. It therefore follows that waves of sound are incapable of polarization; they are, however, capable of interfering with one another and also of being diffracted, since these phenomena do not depend upon the direction of vibration.

169. Equation (9) enables us to calculate the velocity of sound in a gas, and we shall now show how it may be applied to obtain the velocity of sound in air.

We have

$$a = \sqrt{k} = \sqrt{(p/\rho)},$$

where p is the pressure corresponding to a given density. Now it is found by experiment that at 0° C. under a pressure equal to the weight of 1033 grammes per square centimetre, at the place where the experiment is made (i.e. a pressure equal to 1033 g barads[1]), the density of dry air is ·001293 grammes per cubic

[1] In the report of the British Association at Bath, 1888, the Committee on Units recommended the introduction of the following additional units, viz. that

(i) The unit of velocity on the C. G. S. system, i.e. the velocity of one centimetre per second, should be called one *kine*.

centimetre. Hence if we employ the C.G.S. system units, and take $g=981$, we obtain

$$p = 1033\,g = 1033 \times 981, \quad \rho = \cdot 001293,$$

which gives

$$a = 27995;$$

so that the velocity of sound at 0° C. is 279·95 metres per second, or 918·49 feet per second.

The first theoretical investigation respecting the velocity of sound in air was made by Newton, but when his result was submitted to experiment, it was found that it was too small by about one-sixth, inasmuch as the correct result is about 1089 feet per second. This discrepancy between theory and observation was not explained for more than a century, until Laplace pointed out that the use of Boyle's law involved the assumption, that the temperature remains constant throughout the motion, whereas it is well known that when a gas is suddenly compressed its temperature rises. Now it was supposed by Laplace that in the case of waves of sound, the condensation and rarefaction take place so suddenly, that the heat or cold produced have not time to disappear by conduction, and consequently the motion which takes place is much the same as it would be, if the air were confined in a non-conducting vessel. We must therefore ascertain the relation between the pressure and density under these circumstances, and shall accordingly make a short digression on the Thermodynamics of Gases.

Thermodynamics of Gases[1].

170. Let us suppose that a unit mass of gas is contained in a cylinder fitted with a moveable piston, and let p, v, E be its pressure, volume and intrinsic energy. Also let θ be the temperature measured from the absolute zero of the air thermometer, i.e. $-273°$ C.

(ii) The unit of momentum, i.e. the momentum of one gramme moving with the velocity of one kine, should be called one *bole*.

(iii) The unit of pressure, i.e. the pressure of one dyne per square centimetre, should be called one *barad*.

When employing absolute units, it is most important to recollect, that a gramme represents a unit of *mass* and not a unit of *weight*.

[1] The reader is supposed to have studied some elementary work on Thermodynamics, such as Maxwell's *Heat*.

Let a small quantity dH of heat (expressed in mechanical units) be communicated to the gas. If the gas be allowed to expand, the effect of this heat will be (i) to do an amount of work which is equal to pdv, and (ii) to increase the intrinsic energy by dE. Now the first law of Thermodynamics asserts that—*When work is transformed into heat, or heat into work, the quantity of work is mechanically equivalent to the quantity of heat.* It therefore follows from this law, that

$$dE = dH - pdv \quad\text{.......................(11).}$$

By virtue of the laws of Boyle and Charles, the relation

$$pv = h\theta \quad\text{............................(12)}$$

exists between the pressure, volume and temperature of a gas. Any two of the quantities p, v or θ may accordingly be taken as the independent variables. If therefore we take v and θ as independent variables, we may write

$$dH = ldv + K_v d\theta \quad\text{.......................(13).}$$

The quantity l is the *latent heat of expansion*, and K_v is the *specific heat at constant volume*, both expressed in mechanical units.

It is important to notice that the right-hand side of (13) is not a perfect differential; for although dH is *in form* the differential of a quantity dH of heat, yet it is not a definite function of the volume and temperature. The amount of heat communicated to a substance may be measured in mechanical or thermal units, but it cannot be regarded as a function of the state of the substance to which it is communicated.

The intrinsic energy, on the other hand, is a function of the state of the substance, and therefore dE is the differential of a definite function of any two of the quantities p, v, θ.

Equations (11) and (13) are therefore equivalent to

$$dE = (l - p)\, dv + K_v d\theta \quad\text{.................(14).}$$

This equation is true of all substances; but the experiments of Joule and Lord Kelvin have shown that, *the intrinsic energy of a unit of mass of a perfect gas is almost entirely dependent upon its temperature, and not upon its volume.* Accordingly E is a function of θ and not of v, and therefore

$$\frac{dE}{dv} = 0, \qquad \frac{dE}{d\theta} = F(\theta).$$

From these equations combined with (14) we obtain

$$l = p, \quad K_v = F(\theta)$$

which shows that the latent heat of expansion is equal to the pressure, and that the specific heat at constant volume is a function of the temperature.

Equation (13) may therefore be written

$$dH = pdv + F(\theta)\, d\theta,$$

whence by (12)

$$\frac{dH}{\theta} = \frac{hdv}{v} + \frac{F(\theta)}{\theta} d\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

The right-hand side of this equation is a perfect differential of a function which we shall denote by ϕ, accordingly (15) may be written

$$dH = \theta d\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

171. This equation is the analytical expression of a very important but somewhat recondite law, known as the second law of Thermodynamics. For a full discussion of the second law, we must refer to treatises on Thermodynamics, but a few remarks on this subject may be useful.

If one substance at a temperature S be placed in contact with another substance at a lower temperature T, heat will flow from the hot substance into the cold substance; and this process will continue until both substances are reduced to the same temperature. It can however be shown by means of a theoretical heat-engine devised by Carnot, that it is possible to transfer heat from a cold body to a hot body by means of the expenditure of work; and the second law asserts, that it is impossible to do this *without expenditure of work.* The law was first enunciated by Clausius in the following terms:—

It is impossible for a self-acting machine, unaided by external agency, to convey heat from one body to another at a higher temperature.

Lord Kelvin states the law in a slightly different form as follows:—

It is impossible by means of inanimate material agency, to derive mechanical effect from any portion of matter, by cooling it below the temperature of the coldest surrounding objects.

By means of the experimental law, that the intrinsic energy of a gas depends upon its temperature and not upon its volume, the second law of Thermodynamics may be dispensed with in dealing with gases; or to put the matter more correctly, the second law can be deduced as a consequence of the experimental law. But in the case of substances which are not in the gaseous state, the first law is not sufficient to enable us to investigate their thermodynamical properties. Moreover, although it is always assumed that the pressure, temperature and volume are connected together by a certain relation, which may be mathematically expressed by an equation of the form $F(p, v, \theta) = 0$; yet the form of the function F is not accurately known, except in the case of perfect gases. It can be shown that for all substances the second law is mathematically expressed by means of equation (16), and it thus leads to a certain function ϕ, which is capable of being theoretically expressed as a function of any two of the quantities p, v, θ, and which specifies the properties of the substance when it is not allowed to gain or lose heat.

The function ϕ was called the Thermodynamic Function by Rankine; but it is now always known as the *Entropy*.

172. Returning to § 170, let us take p and θ to be the independent variables; equation (13) may then be written

$$dH = Rdp + K_p d\theta,$$

where K_p is the specific heat at constant pressure. Substituting in (11) and eliminating dp by (12) we obtain

$$dE = (Rp/\theta + K_p)\, d\theta - p\,(1 + R/v)\, dv.$$

Since the right-hand side of this equation must be identical with the right-hand side of (14), we must have

$$R = -v, \qquad Rp/\theta + K_p = K_v;$$

whence

$$K_p - K_v = h \quad \text{.........................(17).}$$

Equation (17) shows that the *difference* between the two specific heats is constant; also since the specific heat at constant volume has been shown to be a function of the temperature, it follows that the specific heat at constant pressure must also be a function of the temperature.

173. The value of the specific heat of air at constant pressure has been determined by Regnault, and he finds that it is very nearly independent of the temperature, and is equal to 183·6 foot-

pounds[1] per degree Fahrenheit. It therefore follows from (17) that K_v is also very nearly independent of the temperature.

It also follows from Regnault's experiments, that the value of h for air is 53·21 foot-pounds per degree Fahrenheit; we thus obtain

$$K_v = K_p - h,$$
$$= 183{\cdot}6 - 53{\cdot}21 = 130{\cdot}4.$$

The quantity with which we are most concerned in Acoustics, is the ratio of the specific heat at constant pressure, to the specific heat at constant volume, which is usually denoted by γ. We accordingly find

$$\gamma = K_p/K_v = 1{\cdot}408.$$

The specific heats of all perfect gases are so very nearly independent of the temperature, that they may be treated as constant. The value of the ratio γ, is also approximately the same for all gases.

174. We have already proved the equation $dH = \theta d\phi$. The quantity ϕ is called by Clausius the *entropy of the gas*, and is a quantity which specifies in an analytical form, the properties of a gas which expands or contracts without loss or gain of heat; for when this is the case $dH = 0$, and therefore $\phi = \text{const.}$ If therefore we suppose that ϕ is expressed as a function of p and v, the curve $\phi = \text{const.}$ on the indicator diagram will be a curve which represents the state of the gas under these circumstances. Such curves are called *adiabatic lines*, or *isentropic lines.*

In order to find the form of these curves, we must find an expression for the entropy. Remembering that $l = p$, we obtain from (13) and (16)

$$\theta d\phi = p dv + K_v d\theta \quad\text{.....................(18)},$$
$$= \frac{h\theta}{v} dv + K_v d\theta,$$

whence $$\phi = h \log v + K_v \log \theta + \text{const.} \quad\text{.............(19)}.$$

By (12) and (17), this may be expressed in the form

$$\phi = (K_p - K_v) \log v + K_v \log pv/h + \text{const.},$$

whence $$pv^\gamma = A\epsilon^{\phi/K_v} \quad\text{.........................(20)},$$

where A is a constant.

This is the equation of the adiabatic lines of a perfect gas.

[1] This calculation is taken from Chapter XI. of Maxwell's *Heat*, in which British units are employed.

If ρ be the density of the gas, $v \propto \rho^{-1}$; whence by (20) the relation between the pressure and density of a gas, which expands without loss or gain of heat, is

$$p = k'\rho^{\gamma} \quad\text{.........................(21)},$$

where k' is a constant.

The equation of the isothermal lines may be written

$$pv = (K_p - K_v)\,\theta \quad\text{.........................(22)}.$$

175. The mechanical properties of perfect gases are specified by two quantities, viz. their *densities* and their *elasticities*. The density, as is well known, is defined to be the mass of a unit of volume; but in order to understand what is meant by the elasticity of a gas, some further definitions will be necessary.

The elasticity of a gas under any given conditions, is the ratio of any small increase of pressure, to the voluminal compression thereby produced.

The voluminal compression, is the ratio of the diminution of volume to the original volume.

Hence if v the original volume, be reduced by the application of pressure δp, to $v + \delta v$ (δv being of course negative), the elasticity E is equal to

$$E = -v\frac{dp}{dv} = \rho\frac{dp}{d\rho} \quad\text{.....................(23)}.$$

The quantity E is called the compressibility by Lord Rayleigh (Chapter XV.), and is denoted by him by m.

The value of $dp/d\rho$, and therefore E, depends upon the thermal conditions under which the compression takes place. The two most important cases are, (i) when the temperature remains constant, (ii) when there is no loss or gain of heat. We shall, following Maxwell, denote the elasticity under these two conditions by E_θ and E_ϕ.

In the first case $p = k\rho$, whence $dp/d\rho = k$; accordingly

$$E_\theta = k\rho = p \quad\text{.........................(24)}.$$

In the second case $p = k'\rho^{\gamma}$, whence $dp/d\rho = k'\gamma\rho^{\gamma-1}$; accordingly

$$E_\phi = k'\gamma\rho^{\gamma} = \gamma p \quad\text{....................(25)}.$$

From (24) and (25) we obtain,

$$\frac{E_\phi}{E_\theta} = \gamma = \frac{K_p}{K_v} \quad\text{........................(26)}.$$

Velocity of Sound in Air.

176. Having made this digression upon the thermodynamics of gases, we are prepared to investigate the velocity of sound in a gas.

From (21) we obtain

$$\int \frac{dp}{\rho} = \frac{k'\gamma}{\gamma - 1} \rho^{\gamma - 1}$$

$$= \frac{k'\gamma}{\gamma - 1} \rho_0^{\gamma - 1} + k'\gamma\rho_0^{\gamma - 1} s,$$

since $\rho = \rho_0 (1 + s)$. Whence (4) becomes

$$k'\gamma\rho_0^{\gamma-1} s + U + \dot{\phi} = 0.$$

Eliminating s from (3) we obtain

$$\frac{d^2\phi}{dt^2} = k'\gamma\rho_0^{\gamma-1}\nabla^2\phi - \frac{dU}{dt},$$

and therefore the velocity of sound is equal to $(k'\gamma\rho_0^{\gamma-1})^{\frac{1}{2}}$.

Now $$k = p_0/\rho_0,$$
and $$k' = p_0/\rho_0^{\gamma},$$
whence $$k'\rho_0^{\gamma-1} = k.$$

The velocity of sound is therefore equal to $(k\gamma)^{\frac{1}{2}}$ and is therefore augmented in the ratio $\surd\gamma : 1$. In the case of air, the value of $k^{\frac{1}{2}}$ in feet per second has already been shown to be equal to 918·49, and therefore

$$(k\gamma)^{\frac{1}{2}} = 1083{\cdot}82,$$

which nearly agrees with the value 1089 feet per second given above.

Intensity of Sound[1].

177. We have stated in § 118 that the intensity of sound is measured by the rate at which energy is transmitted across unit area of the wave front. We shall therefore find an expression for this quantity.

Let the velocity potential of a plane wave be

$$\phi = A \cos \frac{2\pi}{\lambda} (x - Vt),$$

[1] Lord Rayleigh, *Theory of Sound*, § 245.

then $$\frac{d\phi}{dx} = -\frac{2\pi A}{\lambda} \sin \frac{2\pi}{\lambda}(x - Vt).$$

If p_0, $p_0 + \delta p$ be the pressures when the air is at rest and in motion respectively, the rate dW/dt at which work is transmitted is

$$\frac{dW}{dt} = (p_0 + \delta p)\frac{d\phi}{dx};$$

and since $$\delta p = -\rho\dot{\phi} = -\rho_0 V \frac{2\pi A}{\lambda} \sin \frac{2\pi}{\lambda}(x - Vt),$$

we obtain

$$\frac{dW}{dt} = -\left\{p_0 - \rho_0 V \frac{2\pi A}{\lambda} \sin \frac{2\pi}{\lambda}(x - Vt)\right\} \frac{2\pi}{\lambda} \sin \frac{2\pi}{\lambda}(x - Vt),$$

$$= \frac{2\pi^2 A^2 \rho_0}{V\tau^2} + \text{periodic terms},$$

since $V\tau = \lambda$.

We therefore see that the rate at which energy is transmitted, consists of two terms: viz. a constant term, which shows that a definite quantity of energy flows across the wave front per unit of time; and a periodic term, which fluctuates in value and contributes nothing to the final effect. The first term measures the intensity of sound, and shows that it varies directly as the square of the amplitude, and inversely as the product of the velocity of propagation in the medium and the square of the period.

CHAPTER XI.

PLANE AND SPHERICAL WAVES.

178. We shall devote the present chapter to the consideration of certain special problems relating to plane and spherical waves of sound.

The theory of the vibrations of strings, which was discussed in Chapter VII., explains the production of notes by means of stringed instruments; but in order to understand how notes are produced by means of wind instruments, it will be necessary to investigate the motion of air in a closed or partially closed vessel. The simplest problem of this kind is the motion of plane waves of sound in a cylindrical pipe, which we shall proceed to consider.

Motion in a Cylindrical Pipe.

179. Let l be the length of a cylindrical pipe, whose cross section is any plane curve, and let the fronts of the waves be perpendicular to the sides of the cylinder.

We shall suppose for simplicity, that the motion is in one dimension, whence measuring x from one end of the pipe, the equation of motion is

$$\frac{d^2\phi}{dt^2} = a^2 \frac{d^2\phi}{dx^2} \qquad (1),$$

where a is the velocity of sound in air.

Since the motion is periodic, we may assume that $\phi = \phi' \epsilon^{\iota nt}$, whence if

$$n/a = 2\pi/\lambda = \kappa \qquad (2),$$

(1) becomes $$\frac{d^2\phi'}{dx^2} + \kappa^2\phi' = 0,$$
the solution of which is
$$\phi' = (A \cos \kappa x + B \sin \kappa x).$$

If the pipe is closed at both ends, $d\phi/dx = 0$ when $x = 0$ and $x = l$; and since
$$\frac{d\phi}{dx} = \kappa (B \cos \kappa x - A \sin \kappa x)\, \epsilon^{\iota nt},$$
the first condition gives $B = 0$, whilst the second condition gives
$$\sin \kappa l = 0,$$
which requires that $$\kappa = i\pi/l,$$
where i is an integer. The value of ϕ in real quantities therefore becomes
$$\phi = A \cos i\pi x/l \cos nt \qquad (3).$$

The wave-length and frequency are thus given by the equations
$$\lambda = 2l/i, \quad n/2\pi = ia/2l \qquad (4).$$

These equations determine the wave-lengths and frequencies of the notes, which can be produced by a pipe of length l, both of whose ends are closed. The frequency of the gravest note is $a/2l$, and its wave-length is $2l$; the frequencies and wave-lengths of the overtones are obtained by putting $i = 2, 3 \ldots$.

180. From (3) we see that $d\phi/dx$ vanishes whenever $x = rl/i$, where r is any integer not greater than i. Corresponding to the ith harmonic, there are therefore $i - 1$ nodes which divide the pipe into i equal parts.

The increment of the pressure due to the wave motion is given by the equation
$$\delta p = -\rho\dot{\phi},$$
and therefore δp vanishes whenever $\cos i\pi x/l = 0$; i.e. whenever $x = (2r + 1)\, l/2i$, where r is zero or any positive integer less than i. Points at which there is no pressure variation are called *loops*. We thus see that corresponding to the gravest note ($i = 1$, $r = 0$), there is a loop at the middle point of the pipe. The loops corresponding to the overtones, occur at points $x = l/2i$, $3l/2i \ldots$; and consequently the loops bisect the distances between the nodes.

The conditions that a node may exist at any point of the pipe, can be secured by placing a rigid barrier across the interior of the

pipe at that point. The conditions for a loop may be approximately realised, by making a communication at the point in question with the external air; and consequently it was assumed by Euler and Lagrange, that the open end of a pipe may be treated as a loop. This supposition is however only approximately true, but the error is small provided the diameter of the pipe is small in comparison with the wave-length. Whenever a disturbance is excited in a pipe which communicates with the air, the external air is set in motion, and a complete solution of the problem would necessitate the motion of the latter being taken into account.

181. Let us in the next place suppose that one end of the pipe is fitted with a disc, which is constrained to vibrate with a velocity $\cos nt$.

The condition to be satisfied at the origin, where the disc is situated, is

$$\frac{d\phi}{dx} = \cos nt, \text{ when } x = 0.$$

If therefore we assume

$$\phi = (A \cos \kappa x + B \sin \kappa x) \cos nt,$$

we obtain

$$B\kappa = 1.$$

If the other end of the pipe is closed, $d\phi/dx = 0$ when $x = l$, whence

$$\phi = \frac{\cos \kappa (l - x)}{\kappa \sin \kappa l} \cos nt.$$

If the other end be open, the condition is that $\phi = 0$ when $x = l$, whence

$$\phi = -\frac{\sin \kappa (l - x)}{\kappa \cos \kappa l} \cos nt.$$

The value of κ is of course n/a.

Reflection and Refraction[1].

182. We shall now investigate the reflection and refraction of plane waves of sound at the surface of separation of two gases.

Let the origin O be in the surface of separation, let the axis of

[1] Green, *Trans. Camb. Phil. Soc.* 1838.

x be drawn into the first medium, and let the axis of z be parallel to the line of intersection of the fronts of the waves with the surface of separation.

Let i be the angle of incidence, r the angle of refraction; also let V, V_1, be the velocities of propagation in the two gases, and ρ, ρ_1 their densities when undisturbed. Then in the first medium we must have

$$\rho' = \rho(1+s),\ p' = k'\rho'^{\gamma} = k'\rho^{\gamma}(1+\gamma s)$$

and in the second medium

$$\rho'_1 = \rho_1(1+s_1),\ p'_1 = k'_1\rho'^{\gamma}_1 = k'_1\rho^{\gamma}_1(1+\gamma s_1).$$

Since the two gases are supposed to be in equilibrium when undisturbed by the waves of sound, we must have

$$k'\rho^{\gamma} = k'_1\rho^{\gamma}_1 \qquad (5).$$

Again, $\quad V^2 = k'\gamma\rho^{\gamma-1},\ V_1^2 = k'_1\gamma\rho_1^{\gamma-1},$

whence $\quad V^2\rho = V_1^2\rho_1 \qquad (6).$

The equations of motion in the first medium are

$$\frac{d^2\phi}{dt^2} = V^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right) \qquad (7),$$

$$\frac{d\phi}{dt} + V^2 s = 0 \qquad (8),$$

and in the second

$$\frac{d^2\phi_1}{dt^2} = V_1^2\left(\frac{d^2\phi_1}{dx^2} + \frac{d^2\phi_1}{dy^2}\right) \qquad (9),$$

$$\frac{d\phi_1}{dt} + V_1^2 s_1 = 0 \qquad (10).$$

The boundary conditions are,

(i) That the component velocity perpendicular to the surface of separation should be the same in both media.

(ii) That the pressure in the two media should be equal at their surface.

The first condition gives

$$\frac{d\phi}{dx} = \frac{d\phi_1}{dx} \qquad (11),$$

and the second gives $\quad p = p_1,$

which by (5), (8) and (10) gives

$$V_1^2\dot{\phi} = V^2\dot{\phi}_1 \qquad (12).$$

If we suppose that the velocity potential of the incident wave is

$$\phi = A\epsilon^{\iota(ax+by+\omega t)} \quad \text{.......................(13)},$$

the velocity potentials of the reflected and the refracted waves may be written

$$\phi' = A'\epsilon^{\iota(a'x+by+\omega t)} \quad \text{......................(14)},$$

$$\phi_1 = A_1\epsilon^{\iota(a_1x+by+\omega t)} \quad \text{......................(15)},$$

for the coefficient of t must be the same in these three equations, because the periods $2\pi/\omega$ of the three waves must be the same; whilst the coefficients of y must be the same, because the traces of the three waves on the surface of separation must move together.

Substituting the value of $\phi + \phi'$ in (7), and the value of ϕ_1 in (9), we obtain

$$\omega^2 = V^2(a^2+b^2) = V^2(a'^2+b^2) = V_1^2(a_1^2+b^2) \quad \text{.........(16)},$$

and therefore $a' = -a$. Also if λ, λ_1 be the wave-lengths in the two media

$$\left.\begin{array}{l} a = (2\pi/\lambda)\cos i,\ b = (2\pi/\lambda)\sin i = (2\pi/\lambda_1)\sin r \\ a_1 = (2\pi/\lambda_1)\cos r,\ \omega = 2\pi V/\lambda = 2\pi V_1/\lambda_1 \end{array}\right\} \text{...(17)}.$$

From the equation $a' = -a$, we see that the angle of incidence is equal to the angle of reflection; and from (17) it follows that

$$\frac{V}{\sin i} = \frac{V_1}{\sin r} \quad \text{........................(18)},$$

which is the law of sines.

To obtain the ratio of the amplitudes, we must substitute the values of $\phi + \phi'$ and ϕ_1 from (13), (14) and (15) in (11) and (12); we thus obtain

$$\left.\begin{array}{l} (A-A')a = A_1a_1 \\ (A+A')V_1^2 = A_1V^2 \end{array}\right\} \text{....................(19)}.$$

By (17) and (18) these become

$$\left.\begin{array}{l} (A-A')\tan r = A_1\tan i \\ (A+A')\sin^2 r = A_1\sin^2 i \end{array}\right\} \text{...............(20)},$$

from which we deduce

$$A' = \frac{A\tan(i-r)}{\tan(i+r)} \quad \text{.....................(21)}.$$

$$A_1 = \frac{2A\sin^2 r\cot i}{\sin(i+r)\sin(i-r)} \quad \text{............(22)}.$$

The first formula is the same as Fresnel's tangent formula for the intensity of the reflected light, when the incident light is polarized perpendicularly to the plane of incidence; and we observe that the reflected wave vanishes when $i + r = \frac{1}{2}\pi$, i.e. when $i = \tan^{-1} V/V_1$.

183. When light is reflected at the surface of a medium, which propagates optical waves with a velocity which is greater than that of the medium from which the light proceeds, it is well known that the light will be totally reflected, when the angle of incidence exceeds a certain value which is called the *critical* angle; and that total reflection is accompanied with a change of phase. We shall now show that a similar phenomenon occurs in the case of sound.

Since $$\cos r = \{1 - (V_1/V)^2 \sin^2 i\}^{\frac{1}{2}},$$
it follows that if $V_1 > V$, $\cos r$ will vanish when $i = \sin^{-1} V/V_1$, and for angles of incidence greater than this value, $\cos r$ will become imaginary; and therefore by (17), a_1 will become a negative imaginary quantity.

When $\cos r$ is imaginary, the values of A' and A_1 given by (21) and (22) become complex, and the formulæ apparently fail. The explanation of this is, that the incident, reflected and refracted waves are the *real* parts of (13), (14) and (15); if therefore A' and A_1 are real, the reflected and refracted waves are given by $A' \cos(-ax + by + \omega t)$ and $A_1 \cos(a_1 x + by + \omega t)$; but if A' is complex, we must put $A' = \alpha + \iota\beta$, and the reflected wave, which is the real part of $(\alpha + \iota\beta)\,\epsilon^{\iota(-ax+by+\omega t)}$, is

$$\alpha \cos(-ax + by + \omega t) - \beta \sin(-ax + by + \omega t)$$
$$= (\alpha^2 + \beta^2)^{\frac{1}{2}} \cos(-ax + by + \omega t + \tan^{-1}\beta/\alpha),$$

which shows that there is a change of phase.

In order to calculate the change of phase, we must put

$$A' = \alpha + \iota\beta,\quad A_1 = \alpha_1 + \iota\beta_1,\quad \mu = V_1/V;$$

also let $$q = (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}/\mu \cos i.$$

From (17) we obtain

$$\frac{a_1}{a} = \frac{\lambda \cos r}{\lambda_1 \cos i} = \frac{V \cos r}{V_1 \cos i} = -\iota q,$$

whence (19) become

$$A - \alpha - \iota\beta = -\iota q(\alpha_1 + \iota\beta_1),$$
$$(A + \alpha + \iota\beta)\mu^2 = \alpha_1 + \iota\beta_1.$$

Equating the real and imaginary parts we obtain

$$\left.\begin{matrix} A - \alpha = q\beta_1, & \beta = q\alpha_1 \\ (A+\alpha)\mu^2 = \alpha_1, & \beta\mu^2 = \beta_1 \end{matrix}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(23),$$

whence

$$\alpha = \frac{A(1-\mu^4q^2)}{1+\mu^4q^2}, \quad \beta = \frac{2A\mu^2q}{1+\mu^4q^2}, \quad \beta_1 = \frac{2A\mu^4q}{1+\mu^4q^2},$$

from which we see that

$$\alpha^2 + \beta^2 = A^2,$$
$$\beta/\alpha = \tan 2e,$$

where $\tan e = \mu^2 q = \mu(\mu^2\tan^2 i - \sec^2 i)^{\frac{1}{2}}$ ....(24).

The reflected wave is therefore

$$\phi' = A\cos(-ax + by + \omega t + 2e),$$

which shows that total reflection takes place, accompanied by a change of phase, whose value is determined by (24).

Since $a_1 = -\iota q a,$

the refracted wave is

$$\phi' = (\alpha_1^2 + \beta_1^2)^{\frac{1}{2}}\epsilon^{qax}\cos(by + \omega t + \tan^{-1}\beta_1/\alpha_1),$$

where $qa = (2\pi/\lambda)(\sin^2 i - \mu^{-2})^{\frac{1}{2}}.$

Since in the second medium x is negative, it follows that the refracted wave is insensible at a distance of a few wave-lengths, and thus the refracted sound rapidly becomes stifled.

Spherical Waves[1].

184. We have already shown that the velocity potential satisfies the equation

$$\frac{d^2\phi}{dt^2} = a^2\nabla^2\phi,$$

where a is the velocity of sound; and if we assume that $\phi = \Phi\epsilon^{\iota\kappa at}$, this becomes

$$(\nabla^2 + \kappa^2)\Phi = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

[1] The remainder of this Chapter is taken from Lord Rayleigh's *Theory of Sound*, Vol. II. Chapter XVII. His original investigations are given in the *Proc. Lond. Math. Soc.* Vol. IV. pp. 93 and 253.

By (12) of § 7, it follows that if r, θ, ω be polar coordinates, the value of ∇^2 is

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2}{d\omega^2};$$

if therefore the motion be symmetrical about the origin, so that Φ is a function of r alone, (25) becomes

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} + \kappa^2\Phi = 0,$$

which may be written in the form

$$\frac{d^2}{dr^2}(r\Phi) + \kappa^2(r\Phi) = 0,$$

the integral of which is

$$\Phi = r^{-1}(A\epsilon^{\iota\kappa r} + B\epsilon^{-\iota\kappa r}) \qquad \text{.................(26).}$$

If the motion is finite at the origin, we must have $A = -B$, in which case

$$\phi = 2\iota A r^{-1}\epsilon^{\iota\kappa a t}\sin\kappa r \qquad \text{.................(27),}$$

in which A may be complex.

185. This equation may be applied to determine the symmetrical vibrations of a gas, which is enclosed within a rigid spherical envelop of radius c; for the condition to be satisfied at the surface of the envelop is

$$d\phi/dr = 0,$$

which gives
$$\kappa\cos\kappa c - c^{-1}\sin\kappa c = 0,$$
or
$$\tan\kappa c = \kappa c \qquad \text{.......................(28).}$$

Since the wave-length $\lambda = 2\pi/\kappa$, and the frequency is equal to $\kappa a/2\pi$, (28) determines the notes which can be produced. The roots of (28) have been investigated by Lord Rayleigh, and he finds that the first root is $\kappa c = 1{\cdot}4303 \times \pi$. We therefore see that the frequency of the gravest note is $\cdot 7151 \times (a/c)$; accordingly the pitch falls as the radius of the sphere increases. This result exemplifies a general law, *that the frequencies of vibration of similar bodies formed of similar materials, are inversely proportional to their linear dimensions.*

The loops are determined by the equation $\sin\kappa r = 0$, which gives $r = m\pi/\kappa$, where m is an integer.

186. Since any circular cone whose vertex is the origin is a nodal cone, the above solution determines the notes which could be produced by a conical pipe closed by a spherical segment of radius c.

If a conical pipe be open at one end, and we assume that the condition to be satisfied at the open end is that it should be a loop, we obtain $\kappa = m\pi/c$, and therefore the value of ϕ is

$$\phi = 2\iota A r^{-1} \epsilon^{\iota m\pi at/c} \sin m\pi r/c.$$

The frequency of the gravest note is therefore $\frac{1}{2}a/c$, which is less than if the pipe were closed.

187. The most general value of ϕ in the case of symmetrical waves is

$$\phi = A r^{-1} \epsilon^{\iota\kappa(at+r)} + B r^{-1} \epsilon^{\iota\kappa(at-r)} \quad \text{............(29)},$$

the first term of which represents waves converging upon the origin, whilst the second represents waves diverging from the origin.

Let us now draw a very small sphere surrounding the origin; then taking the second term of (29), the flux across the sphere is

$$\iint r^2 \frac{d\phi}{dr} d\Omega = - B \iint (1 + \iota\kappa r) \epsilon^{\iota\kappa(at-r)} d\Omega$$
$$= - 4\pi B \epsilon^{\iota\kappa at},$$

when $r = 0$. The second term of (29) therefore represents a source of sound diverging from the pole, of strength $-4\pi B\epsilon^{\iota\kappa at}$; similarly the first term represents a source of sound converging towards the pole[1].

188. The general solution of (25) cannot be effected without the aid of spherical harmonic analysis, but there is one solution of considerable utility, which we shall now consider.

Let $$\phi = \Phi\epsilon^{\iota\kappa at} \cos\theta,$$

where Φ is a function of r alone. Substituting in (25), we obtain

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} - \frac{2}{r^2}\Phi + \kappa^2\Phi = 0.$$

[1] The corresponding problems in two-dimensional motion cannot be investigated without employing the Bessel's function of the second kind $Y_0(\kappa r)$. It is worth noticing, that certain expressions for these functions in the forms of series and definite integrals, can be obtained by means of the theory of sources of sound. See Lord Rayleigh, *Proc. Lond. Math. Soc.*, Vol. XIX. p. 504.

To solve this equation, put $\Phi = dw/dr$ and integrate; we at once obtain

$$\frac{d^2w}{dr^2} + \frac{2}{r}\frac{dw}{dr} + \kappa^2 w = 0,$$

the solution of which has already been shown to be

$$w = r^{-1}(A\epsilon^{\iota\kappa r} + B\epsilon^{-\iota\kappa r});$$

accordingly

$$\Phi = \frac{\iota\kappa}{r}(A\epsilon^{\iota\kappa r} - B\epsilon^{-\iota\kappa r}) - \frac{1}{r^2}(A\epsilon^{\iota\kappa r} + B\epsilon^{-\iota\kappa r}) \quad \text{......(30).}$$

In order to find the condition that the motion should be finite at the origin, we must expand the exponentials in powers of $\iota\kappa r$, and equate the coefficients of negative powers of r to zero; we shall thus find that $A = -B$, whence writing A for $2\iota\kappa^2 A$, the solution becomes

$$\Phi = \frac{A}{\kappa r}\left(\cos\kappa r - \frac{\sin\kappa r}{\kappa r}\right) \quad \text{...............(31).}$$

If gas, contained in a spherical envelop, be vibrating in this manner, the frequency is determined by the equation

$$d\Phi/dr = 0, \text{ when } r = c;$$

which gives
$$\tan\kappa c = \frac{2\kappa c}{2 - \kappa^2 c^2}.$$

The least root of this equation (other than zero), is found by Lord Rayleigh to be $\kappa c = {\cdot}662 \times \pi$; and therefore the frequency of the gravest note is $\cdot 331 \times (a/c)$.

This note is the gravest note which can be produced by gas vibrating within a sphere; it is more than an octave lower than the gravest radial vibration, whose frequency has been shown to be $\cdot 7151 \times (a/c)$.

Since the motion is symmetrical with respect to the diameter $\theta = 0$, every meridional plane is a nodal plane; but since $d\Phi/d\theta$ does not vanish anywhere except along the diameter in question, there are no conical nodal sheets.

189. We shall now consider the motion of a spherical pendulum surrounded with air, which is performing small oscillations.

Since the periods of the pendulum and of the air must be the

same, we may suppose the velocity of the pendulum to be represented by $V\epsilon^{\iota\kappa at}$, and therefore the condition to be satisfied at the surface of the sphere is

$$d\phi/dr = V\epsilon^{\iota\kappa at}\cos\theta \quad\text{.................... (32).}$$

The form of this equation suggests that ϕ must vary as $\cos\theta$; we shall therefore assume that $\phi = \Phi\epsilon^{\iota\kappa at}\cos\theta$, where Φ is given by (30). Since the disturbance is propagated outwards, $A = 0$, and therefore

$$\Phi = -Br^{-2}(1 + \iota\kappa r)\,\epsilon^{-\iota\kappa r}.$$

Substituting in (32), we obtain

$$B = \frac{Vc^3\epsilon^{\iota\kappa c}}{2 - \kappa^2c^2 + 2\iota\kappa c} \quad\text{.................... (33),}$$

where c is the radius of the sphere.

If X be the resistance experienced by the sphere,

$$\begin{aligned} X &= \iint \delta p \cos\theta dS \\ &= -\iint \rho\dot{\phi}\cos\theta dS \\ &= -\tfrac{4}{3}\pi\rho c^2\iota\kappa a\Phi\epsilon^{\iota\kappa at} \\ &= \tfrac{4}{3}\pi\rho c^3 a\dot{\xi}\,\frac{\iota\kappa(1 + \iota\kappa c)}{2 - \kappa^2c^2 + 2\iota\kappa c}, \end{aligned}$$

where $\xi = V\epsilon^{\iota\kappa at}$, is the velocity of the sphere.

Rationalising the denominator, and putting

$$p = \frac{2 + \kappa^2c^2}{4 + \kappa^4c^4}, \quad q = \frac{\kappa^3c^3}{4 + \kappa^4c^4},$$

and remembering that $\ddot{\xi} = \iota\kappa a\dot{\xi}$, we obtain

$$X = M'(p\ddot{\xi} + \kappa aq\dot{\xi}),$$

where M' is the mass of the displaced fluid.

The first term of this expression represents an increase in the inertia of the sphere; whilst the second term represents a resistance proportional to the velocity, which is therefore a *viscous* term, and shows that initial energy is gradually dissipated into space. If M be the mass of the sphere, l the distance of its centre from the point of suspension, the equation of motion of the pendulum is

$$\{M(l^2 + \tfrac{2}{5}c^2) + M'lp\}\,\ddot{\theta} + M'l^2\kappa aq\dot{\theta} + (M - M')\,gl\theta = 0.$$

By § 129, the integral of this equation is of the form

$$\theta = A\epsilon^{-\delta t}\sin(\mu t + \alpha),$$

and the modulus of decay is

$$2\{M(l^2 + \tfrac{2}{5}c^2) + M'l^2 p\}/M'l^2\kappa aq.$$

If the wave-length λ, of the vibrations of the gas, is large in comparison with the radius of the sphere, κc will be of the order c/λ, and will therefore be small; accordingly the value of p will be nearly equal to $\frac{1}{2}$, whilst the value of κq, upon which the viscous term depends, will be of the order c^3/λ^4. We therefore see that in this case the viscous term will be very small, and the motion will die away gradually; hence the sphere will vibrate very nearly in the same manner as if the gas were an incompressible fluid.

If, on the other hand, c were large compared with λ, p would be nearly equal to unity, and the apparent inertia of the sphere would be greater than when c/λ is small; but κq would be of the order c^{-1} and would therefore be small.

190. Another interesting problem is that of the scattering of a plane wave of sound by a fixed rigid sphere, whose diameter is small compared with the wave-length.

Measuring θ from the direction of propagation, the velocity potential of the plane waves may be taken to be

$$\phi = \epsilon^{\iota\kappa(at+x)} = \epsilon^{\iota\kappa(at+r\cos\theta)},$$

the positive sign being taken, because the waves are supposed to be travelling in the negative direction of the axis of x.

If c be the radius of the sphere, it follows that in the neighbourhood of the sphere, κr or $2\pi r/\lambda$ is a small quantity, and therefore expanding the exponential and dropping the time factor for the present, we may arrange ϕ in the form of the series[1]

$$\phi = 1 - \tfrac{1}{6}\kappa^2 r^2 + \iota\kappa r\cos\theta - \tfrac{1}{6}\kappa^2 r^2(3\cos^2\theta - 1)\ldots$$

When the waves impinge upon the sphere, a reflected or

[1] The reader, who is acquainted with Spherical Harmonic analysis, will observe that we have arranged ϕ in a series of zonal harmonics. It can be shown that the solution of (25) can be expressed in a series of terms of the type $F(r)S_n$, where S_n is a spherical surface harmonic.

scattered wave is thrown off, whose velocity potential may be assumed to be

$$\phi' = A_0\Phi_0 + A_1\Phi_1 \cos\theta + \tfrac{1}{2}A_2\Phi_2(3\cos^2\theta - 1) + \dots$$

The quantities Φ_0, Φ_1 are given by (26) and (30) respectively; but since the scattered wave diverges from the sphere, we must put $A = 0$, and take $B = 1$, since the constant B may be supposed to be included in A_0, $A_1\dots$; accordingly

$$\left.\begin{aligned}\Phi_0 &= r^{-1}\epsilon^{-\iota\kappa r}\\ \Phi_1 &= -r^{-2}(1+\iota\kappa r)\,\epsilon^{-\iota\kappa r}\end{aligned}\right\} \quad\dots\dots\dots\dots\dots(34).$$

With regard to Φ_2, it can be verified by trial, that a solution of (25) is $\Phi_2(3\cos^2\theta - 1)$, where Φ_2 is a function of r alone; it will not however be necessary to consider the form of Φ_2, since it introduces quantities of a higher order than $\kappa^2 c^3$, which will be neglected.

The equation to be satisfied at the surface of the sphere is

$$\frac{d\phi}{dr} + \frac{d\phi'}{dr} = 0,$$

when $r = c$. This equation must hold good for all values of θ, whence

$$A_0\frac{d\Phi_0}{dr} - \tfrac{1}{3}\kappa^2 c = 0,$$

$$A_1\frac{d\Phi_1}{dr} + \iota\kappa = 0,$$

which determine A_0, A_1. Substituting from (34) we obtain

$$A_0 = -\frac{\kappa^2 c^3 \epsilon^{\iota\kappa c}}{3(1+\iota\kappa c)} = -\tfrac{1}{3}\kappa^2 c^3,$$

$$A_1 = -\frac{\iota\kappa c^3 \epsilon^{\iota\kappa c}}{2 + 2\iota\kappa c - \kappa^2 c^2} = -\tfrac{1}{2}\iota\kappa c^3$$

approximately, since we shall not retain powers of c higher than c^3. We thus obtain

$$\phi' = -\frac{\epsilon^{-\iota\kappa r}}{r}\left(\tfrac{1}{3}\kappa^2 c^3 - \frac{1+\iota\kappa r}{2r}\,\iota\kappa c^3\cos\theta\right).$$

At a considerable distance from the sphere, the term $\kappa c^3/r^2$ is so small that it may be neglected, we may therefore write

$$\phi' = -\frac{\epsilon^{-\iota\kappa r}}{3r}\,(1 + \tfrac{3}{2}\cos\theta)\,\kappa^2 c^3.$$

Restoring the time factor and putting $\kappa = 2\pi/\lambda$, we finally obtain in real quantities

$$\phi' = -\frac{4\pi^2 c^3}{3\lambda^2 r}(1 + \tfrac{3}{2}\cos\theta)\cos\frac{2\pi}{\lambda}(at - r) \quad \text{.........(35)},$$

corresponding to the wave

$$\phi = \cos\frac{2\pi}{\lambda}(at + x) \quad \text{....................(36)}.$$

Equation (35) accordingly gives the velocity potential of the scattered wave, corresponding to the incident wave whose velocity potential is given by (36). This expression is however only an approximate one, and the correctness of the approximation depends upon the assumption, that the radius of the sphere is so small in comparison with the wave length, that terms of a higher order than c^3/λ^2 may be neglected. We have also neglected $\kappa c^3/r^2$, which is equivalent to supposing, that the point at which we are observing the effect of the scattered wave, is at a considerable distance from the sphere. For a more complete investigation, we must refer to Lord Rayleigh's treatise.

EXAMPLES.

1. If two simple tones of equal intensity and having a given small difference of pitch be heard together, prove that the number of beats in a given time will be greater, the higher the two simple tones are in the musical scale; and prove that the pitch of the resultant sound in the course of each beat is constant.

2. One end of a tube which contains air is open, whilst the other is fitted with a disc, which vibrates in such a manner that the pressure of the air in contact with the disc is

$$\Pi\,(1 - k\sin 2\pi t/\tau)$$

where k is a small quantity. Find the velocity potential of the motion.

3. The radius of a solid sphere surrounded by an unlimited mass of air, is given by $R\,(1 + \alpha\sin nat)$, where a is the velocity of sound in air. Show that the mean energy per unit of mass

of air at a distance r from the centre of the sphere, due to the motion of the latter is

$$\tfrac{1}{4} n^2 a^2 \alpha^2 R^6 (1 + 2n^2 r^2)/r^4 (1 + n^2 R^2).$$

4. Prove that in order that indefinite plane waves may be transmitted without alteration, with uniform velocity a in a homogeneous fluid medium, the pressure and density must be connected by the equation

$$p - p_0 = a^2 \rho_0^2 (\rho_0^{-1} - \rho^{-1}),$$

where p_0, ρ_0 are the pressure and density in the undisturbed part of the fluid.

5. Two gases of densities ρ, ρ_1 are separated by a plane uniform flexible membrane, whose equation is $y = 0$, and whose superficial density and tension are σ and T. If plane waves of sound impinge obliquely at an angle i, and the displacements of the incident reflected and refracted waves of sound and of the membrane, be represented by

(i) $A \sin \{m (x \sin i - y \cos i) - nt + \alpha\}$,

(ii) $A' \sin \{m (x \sin i + y \cos i) - nt + \alpha'\}$,

(iii) $A_1 \sin \{m_1 (x \sin r - y \cos r) - nt + \alpha_1\}$,

(iv) $a \sin (mx \sin i - nt)$,

respectively; find the relations to be satisfied, and prove that the ratio of the intensities of the reflected and incident waves is equal to

$$\frac{(Tm^2 \sin^2 i - \sigma n^2)^2 + (\rho_1 m_1 \sec r - \rho m \sec i)^2}{(Tm^2 \sin^2 i - \sigma n^2)^2 + (\rho m \sec r + \rho m \sec i)^2}.$$

6. If sound waves be travelling along a straight tube of infinite length which is adiathermanous, and no conduction of heat takes place through the air, prove that the equations of motion may be accurately satisfied by supposing a wave of condensation to travel along the tube, with a velocity of propagation which at each point depends only on the condensation at that point, and which for a density ρ is

$$\left[1 + \frac{\gamma + 1}{\gamma - 1}\left\{\left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}(\gamma-1)} - 1\right\}\right]\sqrt{\frac{p_0 \gamma}{\rho_0}},$$

where p_0, ρ_0 are the pressure and density at each end of the wave.

7. Prove that in a closed endless uniform tube of length filled with air, a piston of mass M will perform m complete small vibrations under the elasticity of a spring, if

$$\tan m\pi l/a = \frac{Mm\pi l}{M'a}\left(\frac{n^2}{m^2} - 1\right),$$

where M' is the mass of the air in the tube, and a the velocity of sound, supposing the piston to make n vibrations in a second when the air is exhausted.

8. Investigate the forced oscillations in a straight pipe, which will occur when the temperature of air in the pipe is compelled to undergo small harmonic vibrations expressed by $\theta \cos m(vt - x)$, where x is measured along the axis of the pipe.

9. The greatest angle inclination of the adiabatic lines of a gas to its isothermals occurs, when the slope of the isothermal to the line of zero pressure is $\pi - \cot^{-1}\gamma$; and the locus of all these points of maximum angle, is a straight line through the origin, inclined to the line of zero pressure at an angle $\cot^{-1}\gamma^{\frac{1}{2}}$.

10. A sphere of mean radius R, executes simple harmonic radial vibrations of amplitude α, in air of density ρ; prove that its energy is radiated into the atmosphere in sound waves at the rate

$$2\pi\rho a\left(\frac{a\alpha}{\lambda}\right)^2 \frac{(2\pi R)^4}{(2\pi R)^2 + \lambda^2}$$

per unit of time, where λ is the length of the waves propagated in air, and a is their velocity.

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